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GENERALIZED MULTISETS: FROM ZF TO FSM

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Abstract. We study generalized multisets (multisets that allow possible negative multiplicities) both in the Zermelo-Fraenkel framework and in the finitely supported mathematics. We extend the notion of generalized multiset over a finite alphabet, and we replace it by the notion of algebraically finitely supported generalized multiset over a possibly infinite alphabet. We analyze the correspondence between some properties of generalized multisets obtained in finitely supported mathematics where only finitely supported objects are allowed, and those obtained in the classical Zermelo-Fraenkel framework.

Keywords: Generalized multiset, invariant set, finitely supported mathematics, invariant group, infinite alphabet

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1 INTRODUCTION

Ordinary sets are composed of pairwise different elements which means no two elements are the same. If we accept multiple but finite occurrences of any element we get the notion of multiset which comes to generalize the notion of set. There are many possibilities to define the notion of multiset; the most used procedure is counting the multiplicity of each element. In fact a multiset on Σ is a function from Σ to the set of positive integers $\mathbb{N}$, where each element in Σ has associated its multiplicity. For example, the invariants of a finite abelian group can be represented as a multiset. The prime factorization of a natural number n is another multiset whose elements are primes. Even processes in an operating system can be seen as a multiset, and examples can continue.

Multisets are used in computer science for quantitative analysis and models of resources. References [22] and [23] are the early known references to the applications of multisets in computer science. Multisets and permutations of multisets are applied in a variety of search and sort procedures in [22]. Eilenberg [17] had applied the general theory of multisets to automata. Later, Engelfriet [18] used the multisets to provide a semantical description of some form of the π-calculus. Peterson [31] shows that the very foundation of Petri net theory needs multisets. The algebra of multisets developed in [28] is used in [38] to explicate automated theorem proving for relevance logics, especially in the implementation while using the program KRIPKE. The programming model Gamma [8], where computation can be seen as chemical reactions between data represented as molecules floating in a chemical solution, can be formalized as a multiset rewriting language. Pratt [33] shows how partially ordered multisets (pomsets) can be used to represent parallel processes. He also describes how Petri nets can be modelled as pomsets. Gischer [20] exploits the notion of a partial string and a partial language introduced in [21] to show how pomsets can be used as a model of concurrency. Multisets are also used in database theory [25] or in membrane computing (see [6] and [30]). There are also several attempts to use multisets in programming [10], in describing the evolution of biological systems [27] and new models inspired by cell biology [7]. Rule-based multiset programming paradigm is exploited to study synthetic biology [24]. Basically, multisets are interpreted to represent biological systems, such as molecules in a biochemical system. The evolution of membrane systems is described by employing multisets of objects and multisets of rules. A collection of papers on applications of multisets in computer science can be found in [15] or [36]. A more complete study on multisets (as well as on fuzzy sets, i.e., multisets with real membership) can be found in [39]. The mathematics of multisets have been presented in [4] in the framework of the Fraenkel-Mostowski set theory.

Generalized multisets extend the usual multisets allowing negative multiplicities as well. In a generalized multiset, the multiplicity of an element can be either a positive number, zero, or a negative number. Since the generalized multisets are characterized by the multiplicity of each element, they can also be defined as functions from Σ (the universe of elements) to $\mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. A first study of generalized multisets is due to Blizard [11]. Loeb also investigated generalized mutlisets (see [26]) by using the alternative notion of hybrid set for what we call generalized multiset. However, the first application of the concept of "generalized multiset" is due to Reisig [34] which uses the generalized multisets and the generalized multirelations (which are in fact generalized multisets over the cartesian product $D \times D$ of a set of sorts D) to define relation nets. In [9] generalized multisets are interpreted in a chemical programming framework. In mathematics, an example of the theory of generalized multisets is represented by surreal numbers [16]. Generalized mutlisets could also be used in order to characterize P-systems with anti-matter described in [5]. An algebraic study on generalized multisets in the classical Zermelo-Fraenkel (ZF) framework and in Reverse Mathematics can be found in [1].

Since the experimental sciences are mainly interested in quantitative aspects, and since no evidence exists proving the presence of infinite structures, it becomes useful to study mathematics which deals with a more relaxed notion of infiniteness. The finitely supported mathematics (FSM) introduced in [3] generalizes the classical ZF mathematics, and represents an appropriate framework to work with infinite structures in terms of finitely supported objects. FSM is the mathematics that is inspired by the axioms of the Fraenkel-Mostowski (FM) set theory. The FM set theory has its origins in an approach developed initially by Fraenkel and Mowstowski in 1930s, in order to prove the independence of the axiom of choice and other axioms in the classical ZF set theory. In 2000s, the FM permutation model of Zermelo-Fraenkel with atoms (ZFA) set theory was axiomatized and presented as an independent set theory, named the FM axiomatic set theory [19]. The axioms of the FM set theory are precisely the Zermelo-Fraenkel with atoms (ZFA) axioms over an infinite set of atoms [19], together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x. The original purpose of the FM set theory was to provide a mathematical model for variables in a certain syntax. Since they have no internal structure, atoms can be used to represent names. The finite support axiom is motivated by the fact that syntax can only involve finitely many names. Rather than using a non-standard set theory, one could alternatively work with nominal sets, which are defined within ZF as usual sets endowed with some group actions satisfying a finite support requirement. An alternative definition for nominal sets in the FM framework also exists. They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). These two ways of defining nominal sets finally lead to similar properties (see [3]). According to the previous comment we use the terminology "invariant" for "nominal" in order to establish a connection between approaches in the FM framework and in the ZF framework. Invariant sets represent a tool for describing λ-terms modulo α-conversion [19], automata on data words [13], languages over infinite alphabets [12], or Turing machines that operate on infinite alphabets [14]. The theory of invariant sets provides a balance between rigorous formalism and informal reasoning. This is discussed in [32], where principles of structural recursion and induction are explained within invariant sets.

Actually, FSM represents the ZF set theory rephrased in terms of finitely supported objects. In FSM, we use either 'nominal sets' (which further will be called 'invariant sets') or 'finitely supported sets' instead of 'sets'. Thus, the general principle of constructing FSM is that all the structures have to be invariant or finitely supported.

Translating a ZF result into FSM is not trivial and deserves a special attention. This is because, given an invariant set X, there could exist some subsets of X (and also some relations or functions involving subsets of X) which fail to be finitely supported. Therefore, there may exist some valid results depending on several ZF structures which fail to be valid in FSM if we simply replace "ZF structure" with "FSM structure" in their statement.

The aim of this paper is to develop the theory of generalized multisets both in the ZF framework and in FSM. FSM generalized multisets intend to provide a constructive framework in which we work with sets having finite support. The analogy between the results obtained in FSM and those obtained by using the ZF axioms of set theory is analyzed. The present paper extends [1] and was announced as a future work in [2]. The techniques involved in this paper are similar to those used in [4] where the authors extended the classical multisets by using the theory of nominal sets.

2 PRELIMINARY: INVARIANT SETS

Let A be a fixed infinite (countable or non-countable) ZF set. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$") and if 'ZF' is replaced by 'ZFA' in their statements. Thus, we mention that the theory of invariant sets makes sense both in ZF and in ZFA. Several results of this section are similar to those in [32], but without assuming the set of atoms to be countable.

Definition 1. A *transposition* is a function $(a\,b) : A \to A$ defined by $(a\,b)(a) = b$, $(a\,b)(b) = a$, and $(a\,b)(n) = n$ for $n \neq a, b$. A *permutation* of A is generated by composing finitely many transpositions.

Definition 2. Let S_A be the set of all permutations of A.

1. Let X be a ZF set. An S_A-*action* on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set* is a pair $(X, \cdot)$ where X is a ZF set, and $\cdot : S_A \times X \to X$ is an S_A-action on X.
2. Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ *supports* x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \mid \pi(a) = a, \forall a \in S\}$.
3. Let $(X, \cdot)$ be an S_A-set. We say that X is an *invariant set* if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x. Invariant sets are also called *nominal sets* if we work in the ZF framework [32], or *equivariant sets* if they are defined as elements in the cumulative hierarchy FM_A [19].
4. Let X be an S_A-set and let $x \in X$. If there exists a finite set supporting x, then there exists a least finite set supporting x [19] which is called *the support of* x and is denoted by $supp(x)$. An element supported by the empty set is called equivariant.

Proposition 1. Let $(X, \cdot)$ be an S_A-set and $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported, and $supp(\pi \cdot x) = \pi(supp(x))$.

Example 1.

1. The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$, $\forall \pi \in S_A, a \in A$. Moreover, $supp(B) = B$, $\forall B \subset A$, B finite and $supp(C) = A \setminus C$, $\forall C \subset A$, $A \setminus C$ finite.
2. Any ordinary ZF set X (like $\mathbb{N}$ or $\mathbb{Z}$) is an S_A-set with the trivial S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.
3. If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all $\pi \in S_A$, and all subsets Y of X. For each invariant set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. According to Proposition 1, $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$.
4. Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \star)$ is also an invariant set.
5. The FM cumulative hierarchy FM_A described in [19] (i.e. the universe of all FM sets) is an invariant set with S_A-action $\cdot : S_A \times FM_A \to FM_A$ defined inductively by $\pi \cdot a := \pi(a)$ for all atoms $a \in A$ and $\pi \cdot x := \{\pi \cdot y \,|\, y \in x\}$ for all $x \in FM_A \setminus A$. Any FM set is a finitely supported element in FM_A; additionally an FM set is hereditary finitely supported. An FM set which is empty supported as an element in FM_A is an invariant set.
6. The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is an invariant set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover, $supp(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.

Definition 3. Let $(X, \cdot)$ be an invariant set. A subset Z of X is called *finitely supported* if and only if $Z \in \wp_{fs}(X)$ with the notations of Example 1 (3).

Definition 4. Let X and Y be invariant sets, and let Z be a finitely supported subset of X. A function $f : Z \to Y$ is *finitely supported* if $f \in \wp_{fs}(X \times Y)$.

Proposition 2. Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Let Y^X be the set of all functions from X to Y. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 4 if and only if it is finitely supported with respect the permutation action $\star$.

Proposition 3. [4] Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets, and let Z be a finitely supported subset of X. The function $f : Z \to Y$ is finitely supported in the sense of Definition 4 if and only if there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$.

In order to translate a general ZF result into FSM, one must prove that several structures are finitely supported. Two general methods to prove that a certain structure is finitely supported exist. The first method is a constructive one and it was employed in [2] and [4]: by using some intuitive arguments, we anticipate a possible candidate for the support and prove that this candidate is indeed a support. The second method is based on a general finite support principle which is defined using the higher-order logic.

According to Theorem 3.5 in [32], we have the following equivariance/finite support principle which works over invariant sets.

Theorem 1.

- Any function or relation that is defined from equivariant functions and relations using classical higher-order logic is itself equivariant.
- Any function or relation that is defined from finitely supported functions and relations using classical higher-order logic is itself finitely supported.

3 ZF ALGEBRAIC PROPERTIES OF GENERALIZED MULTISETS

This section represents a survey on the ZF properties of generalized multisets which will be translated into FSM in Section 4. The related results were presented in [1] without proofs.

Definition 5. Given a finite alphabet Σ, any function $f : \Sigma \to \mathbb{Z}$ is called *generalized multiset over* Σ. The value of $f(a)$ is said to be the *multiplicity of* a. The set of all generalized multisets over Σ is denoted by $\mathbb{Z}(\Sigma)$.

The additive structure of $\mathbb{Z}$ induces an additive operation (sum) on generalized multisets in the same way as the additive structure of $\mathbb{N}$ induces an additive operation (sum) on multisets. On $\mathbb{Z}(\Sigma)$ we define an additive law by:

$$\begin{aligned} \text{“} + \text{”} \ &: \ \mathbb{Z}(\Sigma) \times \mathbb{Z}(\Sigma) \to \mathbb{Z}(\Sigma), \\ (f, g) &\mapsto f + g \end{aligned}$$

where $f + g : \Sigma \to \mathbb{Z}$ is defined pointwise by $(f + g)(a) = f(a) + g(a)$ for all $a \in \Sigma$. Since $\mathbb{Z}(\Sigma)$ is formed by all functions from Σ to $\mathbb{Z}$, it is clear that $(\mathbb{Z}(\Sigma), +)$ is an abelian group, the identity being the empty generalized multiset $\theta : \Sigma \to \mathbb{Z}$, $\theta(a) = 0$ for all $a \in \Sigma$, and the inverse of an element $f : \Sigma \to \mathbb{Z}$ is the element $-f : \Sigma \to \mathbb{Z}$ defined by $(-f)(a) = -(f(a))$ for all $a \in \Sigma$. Since $(\mathbb{Z}(\Sigma), +)$ is an abelian group, it follows that $(\mathbb{Z}(\Sigma), +)$ is a $\mathbb{Z}$-module with the scalar multiplication from $\mathbb{Z}$ defined by:

$$\begin{aligned} \text{“} \cdot \text{”} \ &: \ \mathbb{Z} \times \mathbb{Z}(\Sigma) \to \mathbb{Z}(\Sigma), \\ (k, f) &\mapsto k \cdot f \end{aligned}$$

where $k \cdot f : \Sigma \to \mathbb{Z}$ is defined pointwise by $(k \cdot f)(a) = k \cdot f(a)$, for all $a \in \Sigma$ and $k \in \mathbb{Z}$.

Proposition 4. $(\mathbb{Z}(\Sigma), +)$ is a free abelian group.

Proof. If $a \in \Sigma$, we consider the generalized multiset $\tilde{a} : \Sigma \to \mathbb{Z}$ defined by

$$\tilde{a}(b) = \begin{cases} 1 & \text{for } b = a, \\ 0 & \text{for } b \in \Sigma \setminus \{a\}. \end{cases}$$

It is easy to check that every generalized multiset $f \in \mathbb{Z}(\Sigma)$ can be expressed as

$$f = \sum_{a \in \Sigma} f(a) \cdot \tilde{a}.$$

Since Σ is finite, the sum is finite. In fact, the set of generalized multisets $\{\tilde{a} \mid a \in \Sigma\}$ is a basis for the $\mathbb{Z}$-module $\mathbb{Z}(\Sigma)$ since $\{\tilde{a} \mid a \in \Sigma\}$ is also linearly independent. □

According to Proposition 2.37 in [35], two free R-modules are isomorphic iff there are bases of each having the same cardinal, whenever R is a commutative ring. If we denote the basis of $\mathbb{Z}(\Sigma)$ by $\widetilde{\Sigma} = \{\widetilde{a} | a \in \Sigma\}$, it is clear that there is a bijection from Σ onto $\widetilde{\Sigma}$ given by $a \mapsto \widetilde{a}$. It follows that $|\Sigma| = |\widetilde{\Sigma}|$, and so $\mathbb{Z}(\Sigma) \cong FA(\Sigma)$, where $FA(\Sigma)$ represents the free $\mathbb{Z}$-module with basis Σ. $\mathbb{Z}(\Sigma)$ and $FA(\Sigma)$ can be identified (up to an isomorphism).

Since $\mathbb{Z}(\Sigma)$ is the free $\mathbb{Z}$-module with basis $\widetilde{\Sigma}$, it satisfies the universality property described in Proposition 5 (1). We denote by $j : \Sigma \to \mathbb{Z}(\Sigma)$ the function which maps each $a \in \Sigma$ into $\widetilde{a} \in \widetilde{\Sigma}$. It is clear that j is the composition of the standard inclusion $i : \widetilde{\Sigma} \to \mathbb{Z}(\Sigma)$, $i(\widetilde{a}) = \widetilde{a}$ for all $\widetilde{a} \in \widetilde{\Sigma}$ with the bijection of Σ onto $\widetilde{\Sigma}$ defined by $a \mapsto \widetilde{a}$. So, the universality property for $FA(\Sigma)$ can be extended to $\mathbb{Z}(\Sigma)$ by replacing the standard inclusion of Σ into $FA(\Sigma)$ with j; this result is presented as Proposition 5 (2).

Proposition 5.

1. If G is any abelian group and $f : \widetilde{\Sigma} \to G$ is an arbitrary function, then there is a unique homomorphism of abelian groups $g : \mathbb{Z}(\Sigma) \to G$ with $g \circ i = f$, i.e., $g(\widetilde{a}) = f(\widetilde{a})$ for all $\widetilde{a} \in \widetilde{\Sigma}$.
2. If G is any abelian group and $f : \Sigma \to G$ is an arbitrary function, then there is a unique homomorphism of abelian groups $g : \mathbb{Z}(\Sigma) \to G$ with $g \circ j = f$, i.e., $g(\widetilde{a}) = f(a)$ for all $a \in \Sigma$.

Proof. This property can be obtained as a particular case of Proposition 2.34 in [35] proving the universality property for free (left-)modules. □

Some properties of $\mathbb{Z}(\Sigma)$ also follow from the general theory of free modules.

Proposition 6. Let $p : G \to H$ be any surjective homomorphism of abelian groups. For every homomorphism $h : \mathbb{Z}(\Sigma) \to H$ there is a homomorphism of abelian groups $g : \mathbb{Z}(\Sigma) \to G$ such that $p \circ g = h$.

Proof. This result can be obtained as a particular case of Theorem 3.1 from [35] proving that every free (left-)module is projective. The projectivity of $\mathbb{Z}(\Sigma)$ is mathematically expressed as the statement of Proposition 6 in the same way as in the general theory (see [35]). □

Using some basic notions of homological algebra presented in [35] we can give the following result:

Corollary 1. The functor $Hom_{\mathbb{Z}}(\mathbb{Z}(\Sigma), -)$ is an exact functor which means it keeps the exactness of exact sequences.

Proof. The projectivity of a (left-)R-module A is equivalent with the exactness of the functor $Hom_R(A, -)$ (see Proposition 3.2 in [35]). Now, by Proposition 6, $\mathbb{Z}(\Sigma)$ is a projective $\mathbb{Z}$-module, and that means the functor $Hom_{\mathbb{Z}}(\mathbb{Z}(\Sigma), -)$ is an exact functor. □

Theorem 2. If $G \leq \mathbb{Z}(\Sigma)$ is a subgroup of the abelian group $\mathbb{Z}(\Sigma)$ then G is a free abelian group and has a basis of cardinal equal with at most $|\Sigma|$ elements.

Proof. In Theorem 4.13 and Corollary 4.15 in [35] it was proved that, if R is a domain whose all ideals are principal (i.e., cyclic, generated by one element) then every submodule A of a free $R-$module F is also free with $rank(A) \leq rank(F)$. Our proof is complete because all the ideals of $\mathbb{Z}$ are of form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$, and so $\mathbb{Z}$ is a principal ideal domain. □

Definition 6.

1. Adjoin one element to Σ and denote it by 1. A *word on* Σ is either the element 1 or a formal expression $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ where $n \in \mathbb{N}$, $x_i \in \Sigma$ and $\varepsilon_i \in \{\pm 1\}$.
2. Two words are called *equivalent* if one can be obtained from another by repeatedly cancelling or inserting terms of form $x^{-1}x$ or xx^{-1} for $x \in \Sigma$. A word in which all occurring terms can be cancelled is defined to be equivalent with the "empty word". The *equivalence class* of an word w is denoted by $[w]$.
3. The *juxtaposition* of words $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ and $w' = y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}$ is the word $w\#w' := x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}$. Moreover, we define $w\#1 = 1\#w = w$ for all words w.
4. The *free group* $F(\Sigma)$ is the set of all equivalences classes of words on Σ with the group operation $[w] \intercal [w'] := [w\#w']$.

It is easy to check that $\intercal$ is well defined, independent on the chosen representatives, and it verifies the axioms of a group law over $F(\Sigma)$.

Example 2. According to Definition 6, the words of form $aa^{-1}bc^{-1}c$ (for example), $d^{-1}db$ and b are identified (they are in the same equivalence class).

It is worth to note that the order and the multiplicity is important in a word $w = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}$. Another interesting remark is that, if Ω is another alphabet with $|\Sigma| = |\Omega|$ (they have the same cardinal), then $F(\Sigma) \cong F(\Omega)$. The free group on Σ also satisfies the so-called universality property (see Proposition 25.3 in [37]).

Theorem 3. For each group G and each function $f : \Sigma \to G$, there is a unique homomorphism of groups $g : F(\Sigma) \to G$ with $g \circ i = f$, where $i : \Sigma \to F(\Sigma)$ is the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a \in \Sigma$ into the word $[a]$.

Remark 1. The properties of the function g found in Theorem 3 allows us to say that $g(1) = e$ and $g([x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]) = g([x_1])^{\varepsilon_1} \circ g([x_2])^{\varepsilon_2} \circ \dots \circ g([x_n])^{\varepsilon_n} = f(x_1)^{\varepsilon_1} \circ f(x_2)^{\varepsilon_2} \circ \dots \circ f(x_n)^{\varepsilon_n}$ for each word $[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]$; e represents the identity element in G and $\circ$ represents the internal law of G.

We can compare generalized multisets with vectors of integer numbers. It is known that, for $k \in \mathbb{N}$, $k \neq 0$, $\mathbb{Z}^k$ is an abelian group with respect to addition of vectors. Moreover, $\mathbb{Z}^k$ is free with respect to the basis $B = \{e_i = (0,\dots,0,1,0,\dots,0) \mid i = 1,\dots,k\}$. If $\Sigma = \{a_1,\dots,a_k\}$, then $\mathbb{Z}(\Sigma) \cong \mathbb{Z}^k$ as $\mathbb{Z}$-modules and hence as abelian groups.

We can connect all these views using the universal property of the free group $F(\Sigma)$ (it can be connected by this property with any group, and not only with commutative ones as in the case of $\mathbb{Z}(\Sigma)$). If we replace in the statement of Theorem 3, G with $\mathbb{Z}(\Sigma)$, and $f : \Sigma \to G$ with $j : \Sigma \to \mathbb{Z}(\Sigma)$ where j maps each a into $\widetilde{a}$, we get a function $g : F(\Sigma) \to \mathbb{Z}(\Sigma)$ such that $g \circ i = j$, where $i : \Sigma \to F(\Sigma)$ is the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a \in \Sigma$ into the word $[a]$. Now, if $w = [x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]$, then by Remark 1 we obtain that $g(w) = \varepsilon_1 j(x_1) + \varepsilon_2 j(x_2) + \dots + \varepsilon_n j(x_n)$. Now, clearly, g is surjective and, from the first isomorphism theorem for groups, we have $F(\Sigma)/Ker\, g \cong \mathbb{Z}(\Sigma)$.

The Parikh image for multisets [29] has a correspondent for generalized multisets. The generalization is natural. If $\Sigma = \{a_1,\dots,a_k\}$, we define the Parikh image $\varphi_\Sigma : F(\Sigma) \to \mathbb{Z}^k$ in the following way: if $w = [x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]$ then $\varphi_\Sigma(w)$ is the vector in $\mathbb{Z}^k$ whose i^{th} component is $\sum_{\substack{x_j = a_i \\ j=\overline{1,n}}} \varepsilon_j$ for each $i = \overline{1,k}$; if there is no j such that $x_j = a_i$ then the i^{th} component of the vector is defined to be 0. Informally $\varphi_\Sigma(w)$ calculates the number of "occurrences" (even the "negative" ones) of each element from Σ in w. For example, if $\Sigma = \{a,b,c,d,e\}$ and $w = [aa^{-1}bc^{-1}c^{-1}c]$ that is $x_1 = a$, $x_2 = a^{-1}$, $x_3 = b$, $x_4 = c^{-1}$, $x_5 = c^{-1}$, $x_6 = c$, then $\varphi_\Sigma(w) = (1+(-1), 1, (-1)+(-1)+1, 0, 0)$.

If we replace, in the statement of Proposition 5, G with $\mathbb{Z}^k$ and $f : \Sigma \to G$ with the function $\varphi_\Sigma \circ i$ where $i : \Sigma \to F(\Sigma)$ is the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a_u \in \Sigma$ into the word $[a_u]$, then there is a unique homomorphism of abelian groups $\psi_\Sigma : \mathbb{Z}(\Sigma) \to \mathbb{Z}^k$ with $\psi_\Sigma \circ j = \varphi_\Sigma \circ i$, that is $\psi_\Sigma(\widetilde{a_u}) = \varphi_\Sigma([a_u]) =$

$(0,\dots,0,1,0,\dots,0) = e_u$ for all $a_u \in \Sigma$, where $e_u = (0,\dots,0,1,0,\dots,0)$ is the vector in $\mathbb{Z}^k$ whose all components are 0 except the u^{th} component which is 1.

Now, because ψ_Σ maps one-to-one each element from a finite basis of $\mathbb{Z}(\Sigma)$ into an element from a finite basis of $\mathbb{Z}^k$, and $\mathbb{Z}(\Sigma)$ and $\mathbb{Z}^k$ have the same rank, we have that $\psi_\Sigma : \mathbb{Z}(\Sigma) \to \mathbb{Z}^k$ is an isomorphism, and

$$\psi_\Sigma\left(\sum_{i=1}^{k} f(a_i)\cdot \tilde{a}_i\right) = (f(a_1),\dots,f(a_k))$$

for each $f \in \mathbb{Z}(\Sigma)$.

Moreover, the properties of commutative diagrams shows us that $\psi_\Sigma \circ g = \varphi_\Sigma$ where $g : F(\Sigma) \to \mathbb{Z}(\Sigma)$ is the homomorphism built before such that $g \circ i = j$.

$\mathbb{Z}(\Sigma)$	$\mathbb{Z}^k$
$f = \sum_{i=1}^{k} f(a_i)\cdot \tilde{a}_i$	$(f(a_1),\dots,f(a_k))$
multiset addition	vector addition
scalar product	scalar product
θ	$(0,\dots,0)$

Several other order properties of $\mathbb{Z}(\Sigma)$ are presented in [1].

4 FSM GENERALIZED MULTISETS OVER INFINITE ALPHABETS

We formalize now the concept of generalized multisets in FSM. According to Example 1 (2) we already know that $\mathbb{Z}$ is an S_A-set with the S_A-action $\cdot : S_A \times \mathbb{Z} \to \mathbb{Z}$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in \mathbb{Z}$. Also $\mathbb{Z}$ is an invariant set because for each $x \in \mathbb{Z}$ we have that $\emptyset$ supports x. Moreover, $supp(x) = \emptyset$ for each $x \in \mathbb{Z}$.

With the same argumentation as in the case of multisets defined in the FM framework [4], we can extend to notion of generalized multisets by allowing possible infinite alphabets.

Definition 7. Given an invariant set $(\Sigma, \cdot)$ (possible infinite), any function $f : \Sigma \to \mathbb{Z}$ with the property that $S_f \overset{def}{=} \{x \in \Sigma \,|\, f(x) \neq 0\}$ is finite is called *extended generalized multiset over* Σ. The set of all extended generalized multisets over Σ is denoted by $\mathbb{Z}_{ext}(\Sigma)$.

We remark that each function $f \in \mathbb{Z}_{ext}(\Sigma)$ can be expressed as $f = \sum_{a\in\Sigma} f(a)\cdot\tilde{a}$. Since S_f is finite the previous sum is finite. Therefore $\mathbb{Z}_{ext}(\Sigma)$ is a free abelian group with basis $\widetilde{\Sigma}$. Whenever Σ is finite, $\mathbb{Z}_{ext}(\Sigma) = \mathbb{Z}(\Sigma)$.

Proposition 7. Let $(\Sigma, \cdot)$ be an invariant set. We have:

- Each function $f \in \mathbb{Z}_{ext}(\Sigma)$ is finitely supported in the sense of Definition 4. Moreover, $supp(f) \subseteq supp(S_f)$.

- If $f \in \mathbb{Z}_{ext}(A)$, then $S_f = supp(f)$.

Proof. The proof is similar to the proofs of Proposition 6 and Proposition 7 from [4]. We just need to replace $\mathbb{N}$ with $\mathbb{Z}$ in the related proofs. The results are preserved because $\mathbb{N}$ and $\mathbb{Z}$ are endowed with the same S_A-action defined in Example 1 (2). □

Definition 8. An *invariant group* is a triple $(G, \cdot, \diamond)$ such that the following conditions are satisfied:

- $(G, \cdot)$ is a group
- $(G, \diamond)$ is a non-trivial invariant set
- for each $\pi \in S_A$ and each $x, y \in G$ we have $\pi \diamond (x \cdot y) = (\pi \diamond x) \cdot (\pi \diamond y)$ which means the internal law in G is equivariant.

Example 3. The group $(S_A, \circ, \cdot)$ is an invariant group, where $\circ$ is the usual composition of permutations and $\cdot$ is the S_A-action on S_A defined as in Example 1 (6). Since the composition law on S_A is associative, one can easily verify that $\pi \cdot (\sigma \circ \tau) = (\pi \cdot \sigma) \circ (\pi \cdot \tau)$ for all $\pi, \sigma, \tau \in S_A$.

According to Proposition 4 we know that $(\mathbb{Z}(\Sigma), +)$ is a free abelian group if we work in the ZF framework. Analogue $(\mathbb{Z}_{ext}(\Sigma), +)$ is a free abelian group. As in the case of multisets (Theorem 3 from [4]), in FSM we have the following result:

Theorem 4. $\mathbb{Z}_{ext}(\Sigma)$ is a free abelian invariant group whenever $(\Sigma, \cdot)$ is an invariant set. The S_A-action $\star\colon S_A \times \mathbb{Z}_{ext}(\Sigma) \to \mathbb{Z}_{ext}(\Sigma)$ on $\mathbb{Z}_{ext}(\Sigma)$ is defined by $(\pi \star f)(x) = f(\pi^{-1} \cdot x)$ for all $\pi \in S_A$, $f \in \mathbb{Z}_{ext}(\Sigma)$ and $x \in \Sigma$.

Proof. The result follows from Proposition 3.4 in [2]. □

For invariant groups we also have an universality property which is the correspondent of Proposition 5 in FSM.

Theorem 5. Let $(\Sigma, \cdot)$ be an invariant set. Let $j\colon \Sigma \to \mathbb{Z}_{ext}(\Sigma)$ be the function which maps each $a \in \Sigma$ into $\widetilde{a} \in \widetilde{\Sigma}$. If $(G, +, \diamond)$ is an arbitrary abelian invariant group and $\varphi\colon \Sigma \to G$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of abelian groups $\psi\colon \mathbb{Z}_{ext}(\Sigma) \to G$ with $\psi \circ j = \varphi$, i.e., $\psi(\widetilde{a}) = \varphi(a)$ for all $a \in \Sigma$. Moreover, if a finite set S supports φ, then the same set S supports ψ. Therefore, if φ is equivariant, then ψ is also equivariant.

Proof. By refining the original proof from [32] of Theorem 1 we can prove that for any finite set S of atoms, anything that is definable (in the higher-order logic) from S-supported structures using S-supported constructions is S-supported. The requested result in the theorem follows immediately. Alternatively, we can reformulate the direct proof of Theorem 4 from [4] by replacing $\mathbb{N}$ with $\mathbb{Z}$. □

In the previous section we established a connection between $\mathbb{Z}(\Sigma)$ and the free group on Σ denoted with $F(\Sigma)$. A similar result can be proved in FSM.

Theorem 6. $F(\Sigma)$ is an invariant group whenever $(\Sigma, \diamond)$ is an invariant set. The S_A-action $\widetilde{\star} : S_A \times F(\Sigma) \to F(\Sigma)$ on $F(\Sigma)$ is defined by $\pi\widetilde{\star}[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_l^{\varepsilon_l}] = [(\pi \diamond x_1)^{\varepsilon_1}\dots(\pi \diamond x_l)^{\varepsilon_l}]$ for all $\pi \in S_A$ and $[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_l^{\varepsilon_l}] \in F(\Sigma)$.

Proof. The result follows from Theorem 3.6 in [2]. □

Theorem 3 which represents the universality property for $F(\Sigma)$ in the ZF framework has a correspondent in FSM:

Theorem 7. Let $(\Sigma, \diamond)$ be an invariant set. Let $i : \Sigma \to F(\Sigma)$ be the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a \in \Sigma$ into the word $[a]$. If $(G, \cdot, \diamond)$ is an arbitrary invariant group and $\varphi : \Sigma \to G$ is an arbitrary finitely supported function, then there exists an unique finitely supported homomorphism of groups $\psi : F(\Sigma) \to G$ with $\psi \circ i = \varphi$. Moreover, if a finite set S supports φ, then the same set S supports ψ. Therefore, if φ is equivariant, then ψ is also equivariant.

Proof. The result follows from Theorem 3.7 in [2]. The boundedness result claiming that $supp(\psi) \subseteq supp(\varphi)$ follows by making a refinement of Theorem 1 as in the proof of Theorem 5. □

Several results obtained in the previous section (in the ZF framework) can be translated into FSM.

If we replace in the statement of Theorem 7, G with $\mathbb{Z}_{ext}(\Sigma)$, and $\varphi : \Sigma \to G$ with $j : \Sigma \to \mathbb{Z}_{ext}(\Sigma)$ where j maps each a into $\widetilde{a}$, we get an *equivariant* group homomorphism $\psi : F(\Sigma) \to \mathbb{Z}_{ext}(\Sigma)$ such that $\psi \circ i = j$, where $i : \Sigma \to F(\Sigma)$ is the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a \in \Sigma$ into the word $[a]$. Now if $w = [x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]$ then we obtain that $\psi(w) = \varepsilon_1 j(x_1) + \varepsilon_2 j(x_2) + \dots + \varepsilon_n j(x_n)$. Now, clearly, ψ is surjective and, from the first isomorphism theorem for groups we have $F(\Sigma)/Ker\,\psi \cong \mathbb{Z}_{ext}(\Sigma)$. Moreover, in FSM we have the following result:

Proposition 8. $F(\Sigma)/Ker\,\psi$ is an invariant group and the isomorphism Θ between the groups $F(\Sigma)/Ker\,\psi$ and $\mathbb{Z}_{ext}(\Sigma)$, defined by $\Theta(w \top Ker\,\psi) = \psi(w)$ for each $w \in F(\Sigma)$ (where $w \top Ker\,\psi$ is the left coset of w modulo $Ker\,\psi$) is equivariant.

Proof. We remark that Θ is defined as in the standard proof of the first isomorphism theorem for groups. First we prove that we can define an invariant structure on $F(\Sigma)/Ker\,\psi$. We know that $(F(\Sigma), \widetilde{\star})$ is an invariant set (Theorem 6). We define $\odot : S_A \times F(\Sigma)/Ker\,\psi \to F(\Sigma)/Ker\,\psi$ by $\pi \odot (w \top Ker\,\psi) = (\pi\widetilde{\star}w) \top Ker\psi$, for each $w \in F(\Sigma)$ and each $\pi \in S_A$. First we show that $\odot$ is a well defined function. Let $w = [x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}]$ and $v = [y_1^{\delta_1}y_2^{\delta_2}\dots y_m^{\delta_m}]$ be two elements in $F(\Sigma)$ such that $w \top Ker\,\psi = v \top Ker\,\psi$. This means $\psi(w) = \psi(v)$ which by the definition of ψ is the same with $\varepsilon_1 j(x_1) + \varepsilon_2 j(x_2) + \dots + \varepsilon_n j(x_n) = \delta_1 j(y_1) + \delta_2 j(y_2) + \dots + \delta_m j(y_m)$. Now we have $\pi \star (\varepsilon_1 j(x_1) + \varepsilon_2 j(x_2) + \dots + \varepsilon_n j(x_n)) = \pi \star (\delta_1 j(y_1) + \delta_2 j(y_2) + \dots + \delta_m j(y_m))$ for each $\pi \in S_A$ (where $\star$ represents the S_A-action on $\mathbb{Z}_{ext}(\Sigma)$). Since $\mathbb{Z}_{ext}(\Sigma)$ is an invariant group and because j is equivariant (this follows by direct calculation), in the view of Proposition 3 we have $\varepsilon_1 j(\pi\cdot x_1)+\varepsilon_2 j(\pi\cdot x_2)+\ldots+\varepsilon_n j(\pi\cdot x_n)=\delta_1 j(\pi\cdot y_1)+\delta_2 j(\pi\cdot y_2)+\ldots+\delta_m j(\pi\cdot y_m)$ which means $\psi(\pi\widetilde{\star}w)=\psi(\pi\widetilde{\star}v)$ for each $\pi\in S_A$. Therefore, $(\pi\widetilde{\star}w)\top Ker\psi=(\pi\widetilde{\star}v)\top Ker\psi$ for each $\pi\in S_A$ which means that $\odot$ is well defined. Since $\widetilde{\star}$ is an S_A-action on $F(\Sigma)$, an easy calculation shows us that $\odot$ is an S_A-action on $F(\Sigma)/Ker\,\psi$. Moreover, each element in $F(\Sigma)/Ker\,\psi$ is supported by the support of its representative. Therefore, $(F(\Sigma)/Ker\,\psi,\odot)$ is an invariant set. Since $(F(\Sigma),\top,\widetilde{\star})$ is an invariant group (the axioms in Definition 8 are satisfied) it is trivial to check that $(F(\Sigma)/Ker\,\psi,\top,\odot)$ (we denote also with $\top$ the internal law on the factor group $F(\Sigma)/Ker\,\psi$) is an invariant group; the proof is an easy calculation which uses only the definition on $\odot$ and the distributivity property of $\widetilde{\star}$ over $\top$. We claim that Θ is equivariant. For this, in the view of Proposition 3, it is sufficient to prove that for each $\pi\in S_A$ we have $\Theta(\pi\odot(w\top Ker\,\psi))=\pi\star(\Theta(w\top Ker\,\psi))$, $\forall w\in F(\Sigma)$. Let $\pi\in S_A$ be an arbitrary element. Since ψ is equivariant we have $\Theta(\pi\odot(w\top Ker\,\psi))=\Theta((\pi\widetilde{\star}w)\top Ker\,\psi)=\psi(\pi\widetilde{\star}w)=\pi\star\psi(w)=\pi\star(\Theta(w\top Ker\,\psi))$. This means Θ is equivariant. □

If $\Sigma=\{a_1,\ldots,a_k\}$, the Parikh image $\varphi_\Sigma\colon F(\Sigma)\to\mathbb{Z}^k$ is finitely supported. Indeed, $\mathbb{Z}$ is an S_A-set with the S_A-action $\cdot: S_A\times\mathbb{Z}\to\mathbb{Z}$ defined by $\pi\cdot x:=x$ for all $\pi\in S_A$ and $x\in\mathbb{Z}$. From Example 1 (4) we know how an S_A-action on the Cartesian product of two invariant sets looks like. Therefore $\mathbb{Z}^k$ is endowed with a trivial S_A-action defined by $\pi\cdot x:=x$ for all $\pi\in S_A$ and $x\in\mathbb{Z}^k$. Also $\mathbb{Z}^k$ is an invariant set because for each $x\in\mathbb{Z}^k$ we have that $\emptyset$ supports x. Moreover, $supp(x)=\emptyset$ for each $x\in\mathbb{Z}^k$. We prove that $U=supp(a_1)\cup\ldots\cup supp(a_k)$ supports φ_Σ. In the view of Proposition 3 we must prove that we have $\varphi_\Sigma(\pi\widetilde{\star}[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}])=\pi\cdot\varphi_\Sigma([x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}])=\varphi_\Sigma([x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}])$ (because the S_A-action on $\mathbb{Z}^k$ is trivial) for each $\pi\in Fix(U)$ and each $[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}]\in F(\Sigma)$. Indeed if $\pi\in Fix(U)$ we have $\pi\widetilde{\star}[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}]=[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}]$ (Theorem 6) and hence we obtain the relation: $\varphi_\Sigma(\pi\widetilde{\star}[x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}])=\varphi_\Sigma([x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}])$.

If we replace, in the statement of Theorem 5, G with $\mathbb{Z}^k$ and $\varphi\colon\Sigma\to G$ with the function $\varphi_\Sigma\circ i$ where $i\colon\Sigma\to F(\Sigma)$ is the standard inclusion of Σ into $F(\Sigma)$ which maps each element $a_u\in\Sigma$ into the word $[a_u]$, then there exists a unique *finitely supported* homomorphism of abelian groups $\psi_\Sigma\colon\mathbb{Z}(\Sigma)\to\mathbb{Z}^k$ with $\psi_\Sigma\circ j=\varphi_\Sigma\circ i$, that is $\psi_\Sigma(\widetilde{a_u})=\varphi_\Sigma([a_u])=(0,\ldots,0,1,0,\ldots,0)=e_u$ for all $a_u\in\Sigma$.

Definition 9. An *invariant partially ordered set (invariant poset)* is an invariant set $(P,\cdot)$ together with an equivariant partial order relation $\sqsubseteq$ on P.

Loeb's order $\subseteq$ presented in Definition 3.1 from [1] can be naturally extended to $\mathbb{Z}_{ext}(\Sigma)$. The following theorem characterize Loeb's order in FSM.

Theorem 8. If $(\Sigma,\cdot)$ is an invariant set then $(\mathbb{Z}_{ext}(\Sigma),\star,\subseteq)$ is an invariant partially ordered set, where $\subseteq$ represents Loeb's order introduced in Definition 3.1 from [1].

Proof. According to Proposition 7 we have that $(\mathbb{Z}_{ext}(\Sigma),\star)$ is an invariant set with the S_A-action $\star: S_A\times\mathbb{Z}_{ext}(\Sigma)\to\mathbb{Z}_{ext}(\Sigma)$ defined by $(\pi\star f)(x)=f(\pi^{-1}\cdot x)$ for all $\pi\in S_A$, $f\in\mathbb{Z}_{ext}(\Sigma)$ and $x\in\Sigma$. Let $f,g\in\mathbb{Z}_{ext}(\Sigma)$ such that $f\subseteq g$. This means either $f(u) \ll g(u)$ for all $u \in \Sigma$, or $g(u) - f(u) \ll g(u)$ for all $u \in \Sigma$, where $\ll$ is the partial ordering of integers defined as follows: $i \ll j$ iff $i \leq j$ and both of i and j are either at least equal with 0 or smaller than 0. We should prove that $\pi \star f \subseteq \pi \star g$.

Case $f(u) \ll g(u)$ for all $u \in \Sigma$. Let $x \in \Sigma$. We have $(\pi \star f)(x) = f(\pi^{-1} \cdot x) \ll g(\pi^{-1} \cdot x) = (\pi \star g)(x)$. Therefore $(\pi \star f)(x) \ll (\pi \star g)(x)$ for all $x \in \Sigma$, and $\pi \star f \subseteq \pi \star g$.

Case $g(u) - f(u) \ll g(u)$ for all $u \in \Sigma$. Let $x \in \Sigma$. We have $(\pi \star g)(x) - (\pi \star f)(x) = g(\pi^{-1} \cdot x) - f(\pi^{-1} \cdot x) \ll g(\pi^{-1} \cdot x) = (\pi \star g)(x)$. Therefore $(\pi \star g)(x) - (\pi \star f)(x) \ll (\pi \star g)(x)$ for all $x \in \Sigma$, and $\pi \star f \subseteq \pi \star g$. $\square$

Analogue we can prove:

Theorem 9. If $(\Sigma, \cdot)$ is an invariant set then $(\mathbb{Z}_{ext}(\Sigma), \star, \preceq)$ is an invariant partially ordered set, where $\preceq$ is the order on generalized multisets defined in Definition 3.2 from [1].

Proof. According to Proposition 7 we have that $(\mathbb{Z}_{ext}(\Sigma), \star)$ is an invariant set with the S_A-action $\star : S_A \times \mathbb{Z}_{ext}(\Sigma) \to \mathbb{Z}_{ext}(\Sigma)$ defined by $(\pi \star f)(x) = f(\pi^{-1} \cdot x)$ for all $\pi \in S_A$, $f \in \mathbb{Z}_{ext}(\Sigma)$, and $x \in \Sigma$. Let $f, g \in \mathbb{Z}_{ext}(\Sigma)$ such that $f \preceq g$. This means $f(x) \leq g(x)$ for all $x \in \Sigma$. We should prove that $\pi \star f \preceq \pi \star g$. Let $x \in \Sigma$. We have $(\pi \star f)(x) = f(\pi^{-1} \cdot x) \leq g(\pi^{-1} \cdot x) = (\pi \star g)(x)$. Thus, $\preceq$ is equivariant. $\square$

In [2] we proved some embedding theorems for uniform invariant groups (where all the elements are supported by the same set of atoms). We can present a form of the Cayley-theorem for invariant groups which are not necessary uniform. Its proof is similar to the proof of Theorem 5.4 in [2]; we just make the remark that if G is an invariant group, then the set of all finitely supported bijections on G is also an invariant group (the proof is similar to the proof of Proposition 5.3 in [2] and uses Proposition 1).

Theorem 10 (Cayley-theorem for invariant groups)**.** Let $(G, \cdot, \diamond)$ be an invariant group (not necessary uniform). There exists an equivariant isomorphism from G to an invariant subgroup of the invariant group of all finitely supported bijections on G.

Proof. The requested isomorphism is defined as T in the proof of Theorem 5.4 from [2], i.e., for all $g \in G$ we define $T(g)$ as the function $f_g : G \to G$, where $f_g(x) = g \cdot x$, $\forall x \in G$. The definition of T makes sense because, by Proposition 3, $f_g = T(g)$ is supported by $supp(g)$ for each $g \in G$, and so $T(g)$ is a finitely supported bijection on G. The equivariance of T and of the subset $\{T(g) \mid g \in G\}$ follows directly from Theorem 1. $\square$

We are now able to give the Cayley's theorem for $\mathbb{Z}_{ext}(\Sigma)$ in FSM.

Corollary 2. Let Σ be a possible infinite invariant set. There exists an equivariant isomorphism from $\mathbb{Z}_{ext}(\Sigma)$ to an invariant subgroup of the invariant group formed by all finitely supported bijections on $\mathbb{Z}_{ext}(\Sigma)$.

5 CONCLUSION AND FUTURE WORK

FSM represents a useful framework for experimental sciences [3]. Thus, comparing ZF properties of an algebraic structure with its related FSM properties deserves a special attention. The techniques of translating a general ZF result into FSM are presented in [3]. The present paper was announced as a future work in [2]. In this paper we define and study "generalized multisets" both in the ZF framework and in FSM. In Section 3 several ZF algebraic properties of generalized multisets are presented. The results in this section have already been presented in the conference paper [1] without proofs. Some other group theoretical and order properties on generalized multisets can be found in the related reference. However, in this paper we chose to focus only on those ZF results that are also discussed in FSM.

Using similar techniques with those employed for formalizing the multisets in the FM framework [4], we extend generalized multisets over finite alphabets to the framework of invariant sets. We define "extended generalized multisets" over possible infinite alphabets, presenting also some properties of this new concept. The FSM approach allows us to study the generalized multisets over possible infinite alphabets by using a finitary presentation. In Proposition 7 we proved that the set of all extended generalized multisets over an invariant infinite alphabet Σ is an invariant set. Moreover, the set of all extended generalized multisets over Σ is a free abelian invariant group (Theorem 4), and it satisfies the universality property expressed in Theorem 5. The free group over Σ is also an invariant group according to Theorem 6, and it satisfies the universality property presented in Theorem 7. These results are also connected in FSM. The group of all extended generalized multisets can be organized as an invariant partially ordered set (Theorem 8 and Theorem 9). An FSM embedding theorem of Cayley-type is proved for the set of extended generalized multisets over Σ (Corollary 2). Some other order results from [1] can be translated into FSM analogously. However, in order to save space we avoid to present in detail the more laborious calculations.

This paper represents a start point for developing a meta-theory of algebraic structures in FSM. Some similar papers, where algebraic structures are described within nominal sets, are [2] and [4]. However, in [3] we emphasized the differences between the topics generally included into the "FM framework". From now on, we will present our results by employing FSM (which is consistent even over non-countable sets of atoms) instead of nominal sets theory. Our next paper will be focused on studying some similar concepts as in this paper (namely, fuzzy sets, rough sets, Galois connections, and abstract interpretations) in FSM. The techniques involved will be similar to those presented in this paper and in [4]. The goal of another future work will be to study the consistence of choice principles internally in FSM. We will prove that none of the most known choice principles can be formulated in FSM. We will also prove that such a result do not overlap on some known related results from the permutative models of ZFA or from the theory of nominal sets.

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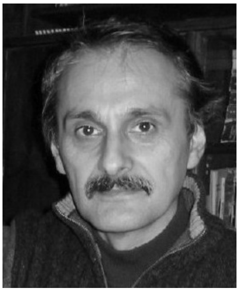

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