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MULTIOBJECTIVE MEMETIC ALGORITHM FOR THE MARKOWITZ MODEL BASED ON INFORMED DECISIONS

Feijoo Colomine Durán

High Performance Computing Laboratory
National Experimental University of Táchira
San Cristóbal, 5001, Venezuela
e-mail: fcolomine@unet.edu.ve

Carlos Cotta, Antonio J. Fernández-Leiva

Instituto de Tecnologías e Ingeniería del Software
Universidad de Málaga, Málaga, 29071, Spain
e-mail: {ccottap, afdez}@lcc.uma.es

Abstract. This paper deals with the selection of investment portfolios with cardinality constraints in the context of the Markowitz model. To this end, a multiobjective memetic algorithm (MA) featuring a novel hybrid combination of population variation based on elite memory and local search is introduced. These heuristic add-ons revolve around the use of the Sharpe index, a return/risk measure used to evaluate the efficiency of the portfolios generated, allowing to focus the search on relevant areas in the nondominated front. The elite memory exploits this indicator to maintain a record of the most effective solutions found and is used as a tunable intensification mechanism. A sensitivity analysis of the algorithm to determine a suitable parameterization is first performed, and then a comparison is made with seven multiobjective evolutionary algorithms. Data from the equity financial market of the Colombian Stock Exchange between 2010 and 2016 are taken as a benchmark and confirm the great potential of the MA for this problem.

Keywords: Memetic algorithms, portfolio optimization, Sharpe index, multiobjective optimization

Mathematics Subject Classification 2010: 68T20, 68U99, 68W50

1 INTRODUCTION

Among the different challenges faced by financial administrators, maintaining profitability and liquidity is a major one, even more so in scenarios of low fixed interest rates that are becoming increasingly prevalent lately. The quest for profitable investments thus goes side by side with an increase in the risk of the investment. As a result, investors will seek to diversify the investments so that money is put on financial instruments driven by different sectors or economic activities and hence subject to different economic cycles (so as to avoid simultaneous losses in all of them, or at least to minimize the chances of such an event). Thus, a portfolio of investments is sought in several financial instruments or sectors, and its composition will be dictated by the degree of risk aversion of the investor [1].

The area of financial management covers a number of theoretical elements and field studies regarding the relationship between risk and performance. This refers to the fact that the best portfolio always depends on the evolution of the market. In general, the investor would seek to maximize the profit or performance of the portfolio while at the same time minimizing the associated risks. Obviously and quite unfortunately, these goals are partially conflicting and several proposals can be found in the literature to address this issue. One of the essential references in this sense is the Markowitz model [2]. It provides a quadratic parameterized model involving a large number of variables whose resolution is not exempt from difficulties from a complexity perspective.

In general, the selection of investments is usually done after acquiring and analyzing a broad data collection characterizing the companies whose assets were selectable. Typical examples of such data are balance sheets and financial statements, business reputation reports, the status of the company within the industry and within the market as a whole, the quality of company management, dividend policy, and many more. This is somewhat simplified within the Markowitz framework, which essentially focuses on selecting investments on the basis of the assumed rational behavior by the investor when the performance of the assets is considered a stochastic process, described by the historical log of returns of the issuing companies (to be precise of three statistical measures of the data: mean, variance, and covariance of return rates). This basically means that investors want to maximize their profit and reject the risk associated with investments. Thus, there naturally emerges the notion of an efficient portfolio as an investment structure that provides the highest possible return given a certain fixed risk, or conversely (and equivalently) one that exhibits the least risk for a given level of profitability. The collection of all such efficient portfolios – those that offer these combinations of maximum profitability (resp. minimum risk) for a given fixed risk (resp. profitability) – is termed the efficient frontier. Once this collection is known, investors can pick their preferred portfolio according to their preferences (one of the criteria that can be used for this purpose is the use of some risk performance indicators, such as the Sharpe index [3]; we will elaborate on this later).

As mentioned above, given a certain risk/profitability profile, the optimal solution in the Markowitz model can be found using quadratic programming. However, in general, this is not the case if this profile is not available a priori or if a broader strategy is sought. This is even more so if the model is enriched with additional constraints and considerations, such as cardinality constraints (bounding the number of investments in the portfolio) or minimum transaction lots, just to cite two examples. In this scenario, a much more complex optimization task is faced, requiring the use of more powerful and flexible algorithmic approaches. In this sense, metaheuristic techniques [4] are an ideal fit. These techniques trade optimality guarantees for cost-effective resolution. Indeed, when adequately crafted, they can provide optimal or near-optimal solutions to a plethora of optimization problems (of continuous, combinatorial, and/or mixed nature).

One of the key issues when metaheuristics are deployed in a given optimization task is precisely the adaptation of the technique to the problem at hand. Such an adaptation is multifaceted and comprises both low-level algorithmic issues (e.g., how solutions are represented and manipulated) and high-level considerations (e.g., how to obtain an adequate search dynamics). These issues lie at the core of some approaches, such as, for example, memetic algorithms (MAs) [5]. From a general perspective, MAs are methods that orchestrate the synergistic interplay of algorithmic components of global search and local search [6, 7]. Very often, global search is attained in MAs via the use of evolutionary algorithms (EAs), whereas local search is conducted using some trajectory-based metaheuristic embedded in the EA.

The main contribution of this work is to propose a novel multiobjective optimization evolutionary algorithm (MOEA) based on a memetic algorithm (constructed as a classical combination of a genetic algorithm and local search) that uses an elite memory built on problem-specific indicators. This multiobjective optimization algorithm has been shown, by experiments, to be efficient when used to deal with a portfolio optimization problem with cardinality constraints. In our proposal, the local search employs specific knowledge provided by the Sharpe index, thus allowing us to focus the global search on the most promising nondominated solutions from the Pareto front. The underlying objective is to align the algorithm with the subsequent decision-making process involving the selection of an appropriate solution from the final Pareto front by a domain expert, by focusing the search on regions that may be more relevant to the decision maker, as opposed to spreading efforts in all directions towards the nondominated front in pursuit of achieving coverage of regions which may be ultimately less relevant. The paper also describes a sensitivity analysis of the parameters of our proposal to provide more information on the insights of our proposed MA. Moreover, we also show that the proposal is efficient compared to the performance of well-known multiobjective evolutionary algorithms that have been applied to solve the problem.

The rest of the paper is structured as follows: first, Section 2 provides an overview of the portfolio optimization problem with cardinality constraints, the Sharpe index, and the techniques that have been proposed in the literature to optimize the problem. Then, our memetic proposal is described in Section 3. An experimental evaluation of our proposal compared to the well-known multiobjective evolutionary algorithms is described in Section 4. The paper ends with the conclusions and points out some lines of future research.

2 PRELIMINARIES

Throughout this section, we shall describe in more detail the precise formulation of the problem considered (Section 2.1), namely optimizing portfolios under cardinality constraints in the context of the Markowitz model, and provide an overview of how the problem has been handled in the scientific literature (Section 2.2), particularly in connection to the main themes of the work, that is, its multiobjective essence and the utilization of metaheuristic approaches of hybrid nature.

2.1 Problem Statement

The model established by Markowitz [2] aims to obtain an optimal portfolio based on the expected profits of the individual financial assets that can be selected, and the risk assumed by the investor when each of these is selected. Both measures are estimated from the previous performance of the assets and can be determined as follows. Firstly, risk is captured by the degree of profitability variation, measured via the covariances of the performance of the assets. The latter are defined as:

$$\sigma_{ij}(\vec{R}) = \sum_{t=1}^{T} \frac{[R_{it} - E(R_i)]\,[R_{jt} - E(R_j)]}{T}, \tag{1}$$

where $\vec{R} = \{R_{it}\}$, $1 \leqslant i \leqslant n$, $1 \leqslant t \leqslant T$, represents the profitability of each asset at each time interval t, $E(R_i)$ is the mean profitability of the i^{th} asset, and T is the number of intervals in the time horizon. This measure is used to estimate the overall risk of the portfolio as a weighted quadratic combination of the covariances of the assets included in it, i.e.,

$$\sigma^2(\vec{R}|\vec{W}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}(\vec{R}), \tag{2}$$

where $\vec{W} = \{w_i\}$, $1 \leqslant i \leqslant n$, is a vector comprising the fraction of the budget allocated to each asset ($0 \leqslant w_i \leqslant 1, \sum_{i=1}^{n} w_i = 1$).

The profitability $E(\vec{R}|\vec{W})$ of a portfolio can be analogously defined as the weighted average of the assets involved, that is

$$E(\vec{R}|\vec{W}) = \sum_{i=1}^{n} w_i E(R_i). \tag{3}$$

Once this information is available, the investor would intuitively look for portfolios for which the profitability is maximal and the risk is minimum, but as it may be evident, these two objectives are partially conflicting in practice. This immediately suggests the intrinsic multiobjective nature of the problem and the existence of a collection of efficient portfolios (i.e., those that maximize the profit for a given risk or minimize the risk for a given profit), that is, those that solve the following bi-objective problem:

$$\left\{\min \sigma(\vec{R}|\vec{W}), \max E(\vec{R}|\vec{W})\right\}. \tag{4}$$

Given this collection (or any approximation of this collection) the investor/expert will pick a certain compromise solution given the desired risk profile of the investment sought. One of the metrics that can be used for this purpose is the Sharpe index [3]. This index determines performance according to the ratio of excess profitability and risk. More precisely,

$$S(\vec{R}|\vec{W}) = \frac{E(\vec{R}|\vec{W}) - R_0}{\sigma(\vec{R}|\vec{W})}, \tag{5}$$

where R_0 is the performance of a portfolio without risk (i.e., for which $\sigma(\vec{R}|\vec{W}) = 0$). Hence, $E(\vec{R}|\vec{W}) - R_0$ can be regarded as excess performance (that is, the extra profit obtained by taking some risks). Sharpe index divides this quantity by the risk of the portfolio (measured as the standard deviation of returns) to obtain an indication of the additional performance that can be expected per unit of risk, so to speak. Obviously, the larger this ratio, the better the portfolio is estimated.

Sharpe index has also an interesting geometrical interpretation. Equation (5) can be rearranged as:

$$E(\vec{R}|\vec{W}) = S(\vec{R}|\vec{W})\sigma(\vec{R}|\vec{W}) + R_0. \tag{6}$$

This is the equation of a straight line in the space (X = risk, Y = profit). This line runs through $(0, R_0)$ – the point corresponding to a portfolio with no risk and hence with performance R_0 – and has a slope equal to a certain fixed Sharpe index. Given a collection of points in the Pareto front defined by Equation (4), the maximum slope of a tangent line to this front that passes through $(0, R_0)$ defines the optimal value of the Sharpe index.

Additionally, consider two portfolios, $\vec{W}_1$ and $\vec{W}_2$. If $\vec{W}_2$ Pareto-dominates $\vec{W}_1$ – meaning that $E(\vec{R}|\vec{W}_1) \leqslant E(\vec{R}|\vec{W}_2)$ and $\sigma(\vec{R}|\vec{W}_1) \geqslant \sigma(\vec{R}|\vec{W}_2)$, where at least one inequality is strict – then $\vec{W}_2$ will have a higher Sharpe ratio. This is because it benefits from a larger numerator and a smaller denominator in Equation (5). This indicates that using the Sharpe ratio in a multiobjective optimizer can help move toward the Pareto front.

2.2 Tackling the Portfolio Optimization Problem (POP)

In light of the complexity and interest of the POP, a lot of attention has been directed towards its resolution. As a result, the literature is rife with diverse complementary approaches. In the following, we provide an overview of some of the most relevant themes in connection with this work, in order to contextualize our further developments.

2.2.1 Algorithms for the POP

Let us begin with classical non-heuristic methods. The Markowitz model, which is the subject of this study, relies on the fact that one of its objective functions (i.e., risk minimization) is a non-parametric quadratic equation. Therefore, to solve this problem, several techniques were initially used to reduce the complexity of the problem. The objective was to convert the problem into a linear programming or non-linear programming problem using models such as Lagrange multipliers or gradient-based approaches to give a satisfactory answer. For example, the use of linear programming techniques using the mean risk model of solvable portfolio optimization with (absolute) Gini coefficient as the risk measure was proposed by Yitzhaki [8]. The mean absolute deviation (MAD) was introduced by Sharpe [9] very early in portfolio analysis. Later, Konno and Yamazaki [10] presented and analyzed a solvable portfolio optimization model based on this risk measure and using linear programming. See [11] for further references on linear programming and other non-metaheuristic techniques applied to portfolio optimization.

Turning now to metaheuristics, these techniques have been among the most widely used to address the problem. For example, Chen et al. [12] proposed an evolutionary model for parameter selection in asset allocation and made a comparison with the Markowitz model for investment portfolio selection. Arriaga and Valenzuela-Rendón [13] in turn proposed a hill climbing method to solve the problem. Furthermore, Rolland [14], used a tabu search algorithm (TS) to find a solution to the problem. In addition to TS, Aldaihani and Al-Deehani [15] developed two mathematical models to solve the problem of maximization of performance and minimization of risk and used TS to test which mathematical model was better. In turn, Crama and Schyns [16], used simulated annealing in the Markowitz model with restrictions, resulting in solutions that were empirically considered quite good.

In addition to trajectory-based metaheuristics such as those mentioned before, metaheuristics based on populations or cooperative behaviors have also been applied to the POP. Focusing first on swarm intelligence [17], an ant colony optimization algorithm incorporating fuzzy outranking preferences was proposed by Fernandez et al. [18]. Meanwhile, Bacanin et al. [19] presented an artificial bee colony algorithm to solve constrained portfolio optimization problems and compared it with two other population-based techniques, such as genetic algorithms and firefly swarm algorithms.

In addition, Deng et al. [20] described a particle swarm optimization algorithm (PSO) which was shown to be more robust and effective than other previously proposed PSOs, especially for low-risk investment portfolios. In turn, Hamdi et al. [21] proposed a PSO algorithm to solve the Mean-CVaR model. PSO showed better results compared to the imperial competitive algorithm in the pricing market of the Iranian automotive industry.

Before moving forward in the discussion of metaheuristic approaches, let us remember at this point that POP is an inherently multiobjective problem when some new constraints are also added such as cardinality, transaction costs, commissions and taxes; therefore, there have been different attempts to address the problem by using multiobjective evolutionary optimization (MOEA) techniques. For example, Chen et al. [22] proposed a multiobjective genetic algorithm (MOGA) model that uses financial knowledge to aid in the selection of optimal investment portfolios. In this proposal, the evaluation criteria were refined with the help of domain-relevant investment knowledge. In this vein, Pai [23] used two metaheuristic strategies from two different genres of evolutionary algorithms, namely, multiobjective differential evolution and multiobjective evolution strategies. These two techniques strategically develop their evolutionary process to solve the investment portfolio optimization problem in multiple stages. Then Drenovak et al. [24] proposed an investment portfolio optimization method with some cardinality constraints and approached its resolution using a parallel optimization framework based on the nondominated sorting genetic algorithm (NSGA-II). Furthermore, Shah and Shah [25] proposed a novel multiobjective genetic programming algorithm (MOGP), which is a memory-enhanced version of the standard SPEA2 algorithm that used memory to store a number of nondominated solutions. These solutions were reused in subsequent stages and a probabilistic metric is employed to choose a Pareto front solution. This is an interesting strategy, to which we will come back later in this work (see also Section 2.2.3).

2.2.2 Hybrid Approaches

Although metaheuristics are general tools for optimization that can be deployed more or less directly on specific problems, there is ample evidence, both empirical and theoretical, that tackling a problem in the absence of knowledge about it and, therefore, without any explicit alignment between the problem and the solver has intrinsic limitations [26]. This underpins the need to exploit problem knowledge to increase optimizer performance. Since it is common for this adaptation to take the form of incorporating problem-specific components within the solver and/or combining ideas from different search approaches, the term *hybrid metaheuristic* [27, 28] is often used in this context.

Needless to say, hybrid metaheuristics have also been applied to POP and therefore we want to make a summary of these approaches and the techniques used. For example, Chen and Chen [29] proposed applying a combination between particle swarm optimization (PSO) and extreme optimization. In the context of an emerging Chinese stock market data set, it was shown that the proposal performed better than a genetic algorithm and also better than a simple PSO. Furthermore, He and Lu [30] studied different algorithmic combinations based on the use of PSO in a fuzzy portfolio model with constraints. As a result, a quantum-behaved PSO algorithm and an improved PSO-based algorithm called the improved quantum-behaved particle swarm optimization algorithm were proposed. In turn, Khan et al. [31] proposed a variant of the beetle antennae search (BAS) method, known as distributed beetle antennae search (DBAS), to optimize multiple portfolio selection problems. DBAS is a hybrid framework and inherits the swarm nature of PSO with the update criteria of BAS.

Memetic algorithms (MAs) can be regarded as a prime example of a hybrid problem-solving strategy. Firstly introduced by Moscato [32] as a framework in which individual search agents alternated between learning phases and periods of collaboration and competition, the most typical MA can be described as a population-based method (which provides global search capabilities and acts as a primary search engine with diversification power) endowed with a local search method (which provides exploitation and intensification); see [5, 33] for a more detailed description. Extending the idea of memetic algorithms to the multiobjective universe, local search techniques have been integrated into multiobjective evolutionary algorithms (MOEAs). The result gives rise to the so-called multiobjective memetic algorithms (MOMA) [34]. The main advantage of adopting this type of hybridization is to accelerate convergence to the Pareto front. However, the use of MOMA presents new problems. One of them is to decide how to select the solutions to which the local search will be applied. It is also nontrivial to determine how long the local search engine should run as its use has an additional computational cost.

In connection with MOMAs, Zhou et al. [35] used NSGA-II in conjunction with the t-SNE algorithm, introduced to reduce the problem of objective redundancy in high-dimensional problems for the Chinese industrial market. In turn, Ferreira et al. [36] proposed a multiobjective optimization model with rebalancing and cardinality constraint to obtain portfolios consistent with the practical aspects of the financial investment process. To perform this optimization, parallel versions of the evolutionary algorithms PDEA, SPEA2, and NSGA-II are developed, seeking a low execution time in accordance with the fast oscillations of the stock market. Zhang et al. [37], described a hybridization mechanism based on hybrid tabu search and differential evolution, which was combined with Pareto ranking theory and empirically shown to outperform NSGA-II.

2.2.3 Using Specific Knowledge and Mechanisms

To improve the optimization process, the use of some prior knowledge or the investor's preferences for decision making have also been considered, sometimes as part of the search process and sometimes as a post-process. For example, within the context of multiobjective optimization, one of the fundamental steps consists in deciding which are the best solutions within the set of nondominated solutions. Several decision-making mechanisms are proposed in the scientific literature along these lines; for example, see [38, 39].

Thus, Zheng and Zheng [40] proposed a multipopulation NSGA-II based on the sparsity strategy to obtain multiple Pareto optimal solutions. Second, in the multi-attribute decision-making stage, the obtained Pareto optimal set is clustered through fuzzy C-means in order to reflect different investment preferences. On the other hand, Ramedani et al. [41] considered the weight of each decision criterion, its impacts, and also the uncertainty in decision making to solve the optimization problem of investment projects. They developed the implementation of a hybrid method based on multiple choice goal programming with utility function and the PSO algorithm. The results showed an improvement in the solution time and quality of the answers of the proposed method. Compared to this line of work, our proposal presents the particularity of using a ranking of portfolios such as Sharpe's so that from it a guided search process can be consolidated.

Another mechanism used for the optimization of the risk-return relationship problem in an investment portfolio is given in the form of using processes to generate some guidance in the search for the Pareto frontier. For example, Fernandez et al. [42] proposed to aggregate multiple criteria based on the particular system of preferences of an investor, producing a selective pressure on the Pareto front towards the portfolio closest to the investor's preferences. In the same vein, Estalayo et al. [43] reported initial findings on the combination of deep learning models and MOEAs to allocate cryptocurrency portfolios. Furthermore, Fernandez et al. [44] presented a new elitist method guided by a genetic algorithm to cope with imperfect knowledge in investor preferences. In turn, Sun et al. [45] described the use of an NSGA-II algorithm with dominance-based elitism features for the investment portfolio allocation problem in the oil industry. They also added additional hybrid interval decomposition prediction methods and introduced optimization parameters and risk preference factors for the best portfolio choice and decision making.

Ruiz et al. [46] analyzed an elite subset of nondominated portfolios that directly matched investor expectations and considered the use of three preference-based multiobjective evolutionary algorithms (WASF-GA, g-NSGA-II, and P-MOGA). In these proposals, repair, mutation, and ad hoc repair operators were used to adapt a knowledge-based search.

Our work presented here is also characterized by hybridization mechanisms and, more precisely, by the integration of local search and an information-based elite memory. The latter is converted into a knowledge base using the Sharpe index, unlike other more generic solutions that are distinguished by the use of elitist memories based on dominance or the use of quality indicators.

3 MATERIALS AND METHODS

In this section, we shall describe our algorithmic proposal based on the use of memetic algorithms and the exploitation of Sharpe index to boost progress towards efficient solutions. This is done in Section 3.2, but previously we will describe the generic algorithmic scenario considered and other multiobjective algorithms that will be used in our experimental comparison (Section 3.1). The data used in the experimentation part will then be shown in Section 3.3.

3.1 Algorithmic Scenario

In light of the scenario described in the previous subsection, the optimization task can be tackled by a multiobjective optimizer that will explore the space of solutions (potential portfolios) aiming to find or approximate the collection of efficient solutions, followed by a selection step in which the Sharpe index is used to extract a solution from this collection. This is the baseline approach against which we will later benchmark our memetic algorithm. Thus, let us start by briefly describing four different well-known MOEAS, which will be later used in the comparative work described in Section 4.

- The *Strength Pareto Evolutionary Algorithm 2* (SPEA2) proposed by Zitzler et al. [47] is an EA that features an external archive of nondominated solutions and a fine-grained strategy to determine the strength of each solution s (the number of solutions dominated by or equal to s, divided by the population size plus one), which will be used for selection (to minimize the strength of solutions that are not dominated by tentative parents). The algorithm also uses a nearest-neighbor density estimation technique to spread the front and a specific archive update strategy.
- The *Nondominated Sorting Genetic Algorithm II* (NSGA-II) proposed by Deb et al. [48] classifies the population into successive nondominated fronts. The nondomination level at which each individual is placed is used as individual rank for selection purposes (via binary tournament on such ranks). The algorithm also uses crowding to perform replacement and spread the Pareto front.
- The *Indicator-Based Evolutionary Algorithms* (IBEA) proposed by Zitzler and Künzli [49] constitutes an attempt to use practical decision-making for Pareto optimization. Unlike the previous two approaches, based on the notion of Pareto dominance, these EAs use some performance indicator to concentrate the search in specific regions of interest in the Pareto front. In this work, we have considered an IBEA based on the ε indicator by Zitzler et al. [50].

- The *Sampling-based HyperVolume-oriented Algorithm* (SHV) and *Hypervolume Estimation Algorithm for Multiobjective Optimization* (HYPE) proposed by Bader and Zitzler [51] are multiobjective EAs based on the hypervolume metric. The algorithms feature a Monte Carlo approach for fast estimation of this metric when the number of dimensions is large, and its use for selection and replacement (in the latter case, nondominated sorting is used to classify the union of parents and offspring; fronts are then successively copied to the next population from best to worst until no more room is available; hypervolume is used to reduce the size of the last front that does not fit in its entirety in the next population).

As can be seen, the above approaches differ in how selection and replacement are done, in order to achieve good coverage and convergence towards the Pareto-optimal front. However, other details of their application to the problem at hand are identical in all cases. Thus, solutions $\vec{W}$ are represented by a sequence of integers $\bar{w}_1, \ldots, \bar{w}_n$, each encoded in binary form. The actual weights of each asset in the portfolio are in the range $[0, 1]$ and can be obtained by normalizing these raw values $w_i = \bar{w}_i / \sum_j \bar{w}_j$. The evaluation is performed by computing the risk and return of the portfolio as described in Section 2.1. In terms of reproduction, we consider standard operators such as two-point crossover and bit-flip mutation. Since a cardinality constraint is enforced, i.e., $\#\{w_i \mid w_i > 0, 1 \leqslant i \leqslant n\} = K$ where $\#S$ indicates the cardinality of set S (in other words, there are exactly K nonzero weights), whenever an infeasible solution is generated, it is discarded.

Once the algorithmic benchmark has been outlined, the next subsection will discuss a memetic approach to portfolio optimization in the Markowitz sense based on the Sharpe ratio as a knowledge engine.

3.2 A Memetic Approach

As highlighted in Section 1, a major challenge in effectively implementing metaheuristics (or any black-box solver for that matter) to a certain optimization task is adapting the former to the latter specific task; cf. [26]. This is exactly one of the core principles of memetic algorithms (MAs) [5]. From a particular perspective, these methods result from merging population-based optimization algorithms – which typically furnish exploration and diversification functions – and a type of local search that enhances the exploitation and intensification of promising areas within the search space. Although this does not exhaust the possibilities within the MA framework – see [7, 52] – it constitutes a basic skeleton onto which a highly effective optimization method can be built. Quite importantly, these methods can be regarded as a complementary problem-solving strategy, intended to benefit from existing algorithmic ideas, combining them in a synergistic way.

In line with this, we consider a memetic approach built around an underlying multiobjective EA by adding two intensifying components to the mix: (1) a memory of elite solutions and (2) a local search method; see Algorithm 1. In this case, the IBEA method described in the previous section has been chosen as the underlying

Algorithm 1 Sharpe-based Memetic Algorithm

Require: μ: population size; λ: number of offspring; p_{LS}: local-search probability; p_{EM}: elite-memory probability

$elite \leftarrow \langle\rangle$
for $i \leftarrow 1$ to μ **do**
    $pop_i \leftarrow$ GenerateSolution()
    UpdateElite($elite$, pop_i)
end for
$\bar{\xi} \leftarrow$ ComputeAvgSharpe($elite$)
while $\neg$ Termination() **do**
    $offspring \leftarrow$ SelectAndReproduce(pop)
    for $i \leftarrow 1$ to λ **do**
        $s \leftarrow$ Sharpe($offspring_i$)
        if $rand < p_{EM}$ **and** $s < \bar{\xi}$ **then**
            $offspring_i \leftarrow$ CorrectSolution($offspring_i$, $elite$)
        end if
        if $rand < p_{LS}$ **and** $s > \bar{\xi}$ **then**
            $offspring_i \leftarrow$ LocalSearch($offspring_i$)
        end if
        UpdateElite($elite$, $offspring_i$)
    end for
    $\bar{\xi} \leftarrow$ ComputeAvgSharpe($elite$)
    $pop \leftarrow$ Replace(pop, $offspring$)
end while

evolutionary search engine due to its good performance in this problem domain [53]. As for the intensifying components, their function and purpose are as follows.

To begin with, the memory of elite solutions is a ranked list of the best solutions (with respect to their Sharpe index) that have been generated at any particular moment. The list starts out empty, with a fixed maximum capacity of Θ. Every time a new solution is generated, its Sharpe index is generated. If it is positive, then it is tested against this elite memory: If there is room in it, it is inserted; otherwise, it substitutes the worst solution (according to the Sharpe index) in the collection, provided the former is better than the latter. The purpose of this list is to provide additional search intensification. To this end, the algorithm uses in each iteration the average Sharpe index of elite memory $\bar{\xi}$: for every new individual created in that generation, a correction procedure is triggered with certain probability p_{EM}: If the Sharpe index of the new individual is below $\bar{\xi}$ it is replaced by a random individual from the elite solutions memory. By doing so, below-average solutions are discarded, and the search refocuses around previous elite solutions. It must be noted that this procedure is intended to work in conjunction with the stochastic local search procedure, and thus, rather than merely exploiting good genetic material, it provides a promising starting point for performing local adjustment.

Algorithm 2 LOCALSEARCH (p)

Require: p: the starting solution

1: $\xi \leftarrow$ SHARPE(p)
2: $i \leftarrow \max_{\mathcal{N}}$
3: **while** $i > 0$ **do**
4:     $p' \leftarrow$ GENERATENEIGHBOR(p)
5:     $\xi' \leftarrow$ SHARPE(p')
6:     **if** $(\xi' > \xi)$ **then**
7:         $p \leftarrow p'$; $\xi \leftarrow \xi'$
8:         $i \leftarrow i - 1$
9:     **else if** $\frac{\xi - \xi'}{\xi} > \theta_\xi$ **then**
10:         $i \leftarrow i - \eta_1$
11:     **else**
12:         $i \leftarrow i - \eta_2$
13:     **end if**
14: **end while**
15: **return** p

Let us now turn our attention to the second addition, namely the local search component, which is here incarnated in a variant of a first-ascent hill-climbing method. The application of this local search method is shown in line 15 of Algorithm 1 and is detailed in Algorithm 2. The method receives a solution to improve if the random test is satisfied with a probability P_{LS} and the solution has a Sharpe index above $\bar{\xi}$ (to focus the search on promising solutions only) and iterates a procedure consisting of generating a neighbor (mutating a weight) and accepting as the current solution if it has a better Sharpe index. This is done until a certain fixed computational budget allocated to this component is exhausted (the local search routine has an acceleration mechanism as shown in lines 9–12, whereby if the neighbor is worse than the current solution, the computational budget is reduced to avoid unfruitful computations; two different penalties are considered depending on whether the neighbor is worse by more than a certain threshold θ_ξ or not).

Notice how the MA uses a partial Lamarckianism scheme [54] by having a local search activated with a certain probability (hence aiming to reduce unproductive computational efforts resulting from the indiscriminate application of local search to every solution). The source code of the algorithm is available in our code repository[1].

3.3 Data Used in the Analysis

This work has considered the monthly closing prices in the Colombian Stock Market of 20 equity funds during the years 2010 to 2016. Usually, it is convenient to pick an interval of at least 5 years, so as to capture, in an ideal scenario, a complete

[1] `https://github.com/fcolomine/MAsharpe`

market cycle with the corresponding fluctuation in stock prices. Indeed, Markowitz model works under the assumption that all relevant information is reflected in asset prices, so that information provided by the historical data allows estimating future returns and risk in a reliable way. However, it is not advisable to pick a very large interval (e.g., 10 years or larger), since the actual status of the fund may not be well represented by very old market information. In this sense, our dataset fits comfortably within these limits.

Data have been collected from the public website [55]. Table 1 describes the funds considered[2]. We consider these data to be representative of the behavior of these equity funds. Furthermore, notice how this dataset incorporates assets from different sectors (infrastructure, financing, energy, etc.) and with different trading behavior, hence being a representative market sample that allows investment diversification.

	ECOPETROL	PREC	PFBCOLOM	GRUPOSURA	GRUPOAVAL
Profitability	−0.00429493	−0.02689857	0.00827647	0.00794438	0.011147021
Risk	0.00671900	0.01193778	0.00170523	0.00161020	0.00243122
	CORFICOLCF	EXITO	ISAGEN	ISA	PFCORFICOL
Profitability	0.014101804	0.010228074	0.005079423	−0.001401204	0.013321116
Risk	0.00152303	0.00222107	0.00089910	0.00205940	0.00138877
	BCOLOMBIA	CEMARGOS	EEB	BOGOTA	BVC
Profitability	0.00750252	−0.003115673	−1,782468658	0.013590999	−0.001949743
Risk	0.00179756	0.00269953	0.15645018	0.00148398	0.00277334
	VALOREM	PROTECCION	FABRICATO	COLTEJER	ETB
Profitability	0.000381456	0.008730646	−0.033799508	−0.031747288	−0.007452069
Risk	0.00032588	0.00010042	0.00509110	0.00187508	0.00055756

Table 1. Funds from the BVC included in the dataset used in the experimentation. For each fund the profitability ($E(R_i)$) and risk ($\sigma(R_i)$) in the period 2010-2016 are shown.

4 EXPERIMENTATION

In this section we are going to capture the entire experimentation process developed according to the framework defined in Section 3.3. To this end, we shall first describe the experimental setting and parameterization of our experiments in Section 4.1. Then, we will conduct an extensive experimentation that includes both a sensitivity analysis of our MA in order to calibrate its behavior and observe how the search process varies in response to adjustment in the parameter involved, and a comparison with other MOEAs. This is done in Section 4.2.

[2] These data are available in our data repository at `https://osf.io/wg7mn/`

4.1 Experimental Setting

Experiments have been carried out with the five algorithms described in Section 3.1 – namely NSGA-II, SPEA2, IBEA, SHV and HYPE – and with our MA described in Section 3.2. We have used the PISA library (a Platform and Programming Language Independent Interface for Search Algorithms) proposed by Bleuler et al. [56] for the basic multiobjective EAs, as well as for the MA evolutionary engine. In all cases, the weights are encoded using 10 bits (which results in a bitstring length $\ell = 200$, since there are 20 funds as indicated in Section 3.3), the crossover rate is $P_c = 0.8$, the mutation rate is $P_m = 1/\ell$ and the population size is 2ℓ. Regarding the MA, the memory of elite solutions has a maximum size $\Theta = 30$, the correction probability is P_{EM}, the local search probability is P_{LS}, and the maximum number of neighbors explored is $\max_{\mathcal{N}} = 10$. Other local search parameters are $\theta_\xi = 0.02, \eta_1 = 4, \eta_2 = 2$. The algorithms have been run for a maximum number of 20 000 fitness evaluations (except the MA, which is reduced to 18 500 to account for the overhead of local search), and 30 runs are performed for each of them and each value of K (we have considered $K \in \{14, 16, 18, 20\}$ in these experiments). These values were chosen for clarity in the presentation of the results and are also largely consistent with financial theories.

In order to evaluate the results provided by each method, we consider a twofold perspective, namely the quality of the Pareto front obtained and the quality of the selected portfolio from this front. For the latter aspect, we consider the Sharpe index as mentioned in Section 2.1. Regarding the evaluation of Pareto fronts, we have considered two well-known performance indicators: the hypervolume indicator [57] and the generational distance GD indicator [58]. The first indicates the region in the fitness space dominated by the front (and thus the larger the better). To measure this, a reference point is required to delimit the fitness space, which in this case is obtained by taking the maximum risk and the minimum benefit attained in solutions on the best-known Pareto front (the combined Pareto front for the particular value of K considered, taking the fronts of all algorithms included in any specific comparison). As to the second indicator, it estimates how closely a certain front approximates another one (the true Pareto-optimal front if known, or a reference front otherwise). We have considered the unary version of this indicator, again taking the combined Pareto front for the algorithms considered in the comparison as a reference set. Since this is a measure of distance to the reference set, lower values of GD are better. In both cases, the absolute values of the indicators are normalized. To do so, a counterreference point is determined as the hypothetical ideal point (maximum observed performance and minimum observed risk in the combined Pareto front). Notice that this counter-reference point and the reference point for the hypervolume determine together the minimum rectangle that contains the combined Pareto front. The hypervolume values are then expressed as a fraction of the volume of this rectangle and the GD values are expressed as a percentage of the diagonal of this rectangle.

4.2 Experimental Results

The experimentation has been structured in two phases[3]. In the first, we perform a sensitivity analysis to identify the range of values in which the MA yields better performance; see Section 4.2.1. After having done this, we embark on the second phase – see Section 4.2.2 – in which we compare the MA with different evolutionary multiobjective optimizers (cf. Section 3.1) using the indicators suggested in Section 4.1.

4.2.1 Sensitivity Analysis

In order to find the best parameterization, we consider three different stages in which the MA features different subsets of components, namely

1. local search only,
2. elite memory only, and
3. local search plus elite memory.

In each case, we statistically analyze the numerical results and perform a comparative analysis to identify the parameter regions of interest. We will use the notation $\mathrm{MA}_{x,y}$ to denote an MA with $P_{LS} = x$ and $P_{EM} = y$ (note that $\mathrm{MA}_{0,0}$ would reduce to IBEA).

Local Search. Herein, we consider the MA is only using the local search component. Thus, the size of the elite memory is $\Theta = 0$ and $P_{EM} = 0$. We analyze the impact of the probability of application of local search P_{LS}, and to this end we consider values $P_{LS} \in \{0, 0.05, 0.10, 0.20, 0.50, 0.80, 1.00\}$ for this parameter. The numerical results for the two multiobjective performance indicators (hypervolume and generational distance) and portfolio efficiency (measured by the Sharpe index) are shown in Table 2 along with a statistical significance analysis. To be precise, in this experiment and all subsequent ones we have taken as a reference the parameter configuration with the best median results for each of the performance indicators and conducted a Wilcoxon Ranksum test, reporting the so-obtained significance values into three tiers, namely $\alpha \in \{0.01, 0.05, 0.1\}$.

As can be seen, the results tend to improve for each performance indicator when the local search probability is increased. Although the general trend may seem intuitively expected, it is interesting to note that partial Lamarckianism is often overlooked, even though it can be an appropriate strategy in some domains [59]. Hence, it is worth obtaining empirical evidence of the benefits of high local search rates and the corresponding increase in search intensification in this particular domain. This is even more so taking into account the multiobjective nature of the

[3] All experimental data generated are available in our data repository at `https://osf.io/uekt6/`.

	$K = 14$														
	Sharpe index					hypervolume					GD				
P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.408	0.407	0.000	0.073 %	■	0.810	0.802	0.002	0.269 %	■	0.487	0.636	0.112	17.119 %	■
0.05	0.408	0.406	0.000	0.094 %	■	0.811	0.802	0.003	0.431 %	■	0.422	0.632	0.075	11.624 %	■
0.10	0.408	0.407	0.000	0.084 %	■	0.815	0.808	0.003	0.384 %	■	0.344	0.520	0.094	17.893 %	■
0.20	0.408	0.408	0.000	0.077 %	■	0.816	0.810	0.001	0.176 %	■	0.319	0.445	0.050	11.657 %	■
0.50	0.408	0.407	0.000	0.085 %	■	0.817	0.812	0.002	0.246 %	●	0.258	0.394	0.046	11.202 %	●
0.80	0.408	0.408	0.000	0.073 %	●	0.819	0.813	0.002	0.229 %	⋆	0.238	0.349	0.057	15.850 %	⋆
1.00	0.409	0.408	0.000	0.051 %	⋆	0.818	0.811	0.002	0.197 %	●	0.244	0.395	0.047	11.805 %	●

	$K = 16$														
	Sharpe index					hypervolume					GD				
P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.406	0.405	0.001	0.163 %	■	0.812	0.804	0.005	0.679 %	■	0.791	1.092	0.157	14.382 %	■
0.05	0.407	0.405	0.001	0.143 %	■	0.820	0.810	0.003	0.412 %	■	0.556	0.888	0.158	17.214 %	■
0.10	0.407	0.406	0.000	0.112 %	■	0.824	0.815	0.002	0.269 %	■	0.453	0.696	0.081	11.197 %	■
0.20	0.407	0.406	0.000	0.104 %	■	0.825	0.816	0.003	0.365 %	■	0.456	0.680	0.098	13.770 %	■
0.50	0.408	0.407	0.000	0.105 %	●	0.829	0.820	0.003	0.308 %		0.304	0.555	0.086	15.730 %	
0.80	0.408	0.407	0.000	0.081 %	■	0.826	0.819	0.003	0.351 %	○	0.393	0.554	0.059	10.362 %	
1.00	0.408	0.407	0.000	0.059 %	⋆	0.827	0.822	0.002	0.261 %	⋆	0.380	0.515	0.060	11.721 %	⋆

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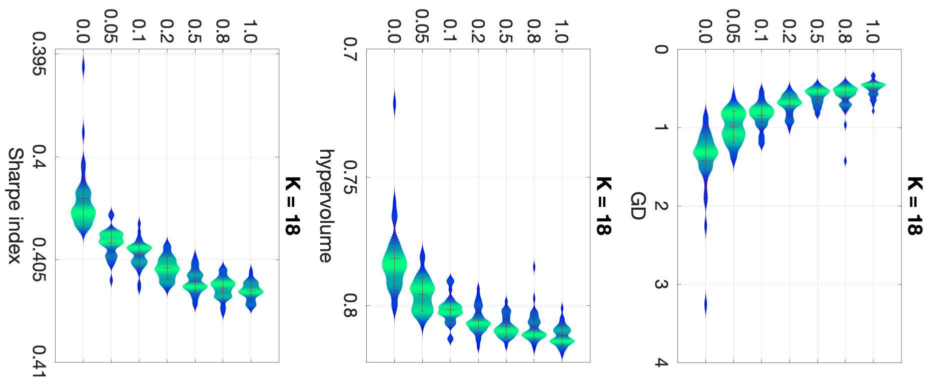
	$K = 18$														
	Sharpe index					hypervolume					GD				
P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.404	0.402	0.001	0.150 %	■	0.800	0.784	0.006	0.758 %	■	0.862	1.333	0.160	11.844 %	■
0.05	0.406	0.404	0.000	0.120 %	■	0.807	0.795	0.005	0.588 %	■	0.604	1.014	0.143	14.423 %	■
0.10	0.406	0.405	0.000	0.102 %	■	0.813	0.802	0.002	0.277 %	■	0.520	0.829	0.100	11.915 %	■
0.20	0.406	0.405	0.000	0.098 %	■	0.816	0.807	0.003	0.410 %	■	0.495	0.691	0.087	12.353 %	■
0.50	0.407	0.406	0.000	0.081 %	■	0.816	0.809	0.003	0.377 %	■	0.437	0.580	0.082	13.539 %	■
0.80	0.408	0.406	0.000	0.095 %		0.817	0.811	0.003	0.362 %	●	0.422	0.578	0.094	15.536 %	■
1.00	0.407	0.407	0.000	0.066 %	⋆	0.819	0.813	0.003	0.314 %	⋆	0.331	0.492	0.066	12.811 %	⋆
	$K = 20$														
	Sharpe index					hypervolume					GD				
P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.398	0.389	0.007	1.714 %	■	0.808	0.760	0.032	4.256 %	■	2.142	3.577	1.229	28.962 %	■
0.05	0.403	0.395	0.008	2.008 %	■	0.835	0.790	0.040	5.120 %	■	1.299	2.684	1.354	41.879 %	■
0.10	0.402	0.400	0.002	0.482 %	■	0.845	0.816	0.016	1.922 %	■	0.938	1.653	0.528	27.191 %	■
0.20	0.404	0.401	0.002	0.377 %	■	0.848	0.824	0.012	1.509 %	■	0.917	1.524	0.371	22.919 %	■
0.50	0.406	0.403	0.001	0.242 %	■	0.860	0.837	0.007	0.872 %	■	0.586	1.096	0.247	21.710 %	■
0.80	0.407	0.405	0.000	0.108 %		0.859	0.845	0.007	0.888 %		0.521	0.878	0.200	21.589 %	○
1.00	0.406	0.405	0.001	0.156 %	⋆	0.863	0.849	0.008	0.892 %	⋆	0.490	0.748	0.177	22.204 %	⋆

Table 2. Numerical results using local search. In this table and in subsequent ones, the best (x^*), median ($\tilde{x}$), quartile deviation (QD) and coefficient of quartile deviation (QCD) are shown for each performance indicator. The best configuration for each indicator is marked by $\star$; the remaining configurations are marked with symbols that indicate statistically significant difference at $\alpha = 0.1$ (○), $\alpha = 0.05$ (●), and $\alpha = 0.01$ (■). Absence of a symbol indicates no statistically significant difference with respect to the $\star$ configuration.

problem, whereby local search could have different effects on different performance indicators (recall that the local search component is based on maximizing the Sharpe index). In this sense, note in Table 2 how the highest rate of local search produces the best performance in terms of the Sharpe index, but not necessarily in terms of the other performance indicators in the case of the most constrained scenario ($K = 14$), where high but not maximum values of the local search parameter (around $P_{LS} = 0.8$) seem to provide the best results. This said, the increased rates of local search generally correlate with better performance in the remaining scenarios (see, e.g., Figure 1). This can also be seen in Figure 2, where a greater intensification of the search in the Pareto region associated with a greater portfolio risk can be easily observed.



a) Sharpe index b) Hypervolume c) Generational distance

Figure 1. Distribution of the performance indicators attained across 30 runs of the MA with different parameterization of local search and no elite memory for $K = 18$

Elite Memory. After having studied local search, we now turn our attention to the use of elite memory. Similarly to what was done in the previous subsection, we consider our basic MA with no local search ($P_{LS} = 0$) and different settings for the probability of application of elite memory P_{EM}, namely $P_{EM} \in \{0, 0.05, 0.10, 0.20, 0.50, 0.80, 1.00\}$. In all cases, the size of the elite memory is $\Theta = 30$. Table 3 provides the numerical results obtained, and Figure 3 provides a comparative representation of performance as before.

The results obtained follow an analogous pattern to that observed for the local search and even depict a more marked difference between the effects on either performance indicator. Intuitively, the probability of application of elite memory provides a trade-off between diversification and intensification. Although high values of P_{EM} exert a high pressure on a very localized region of the search space, low values of P_{EM} result in a milder pressure extended over a more diverse set of solutions. In

	$K = 14$														
	Sharpe index					hypervolume					GD				
P_{EM}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.408	0.407	0.000	0.073 %	■	0.815	0.808	0.002	0.261 %	●	0.512	0.722	0.107	14.779 %	
0.05	0.408	0.407	0.000	0.082 %	■	0.816	0.809	0.002	0.244 %	○	0.492	0.733	0.111	15.114 %	
0.10	0.407	0.406	0.000	0.083 %	■	0.813	0.802	0.004	0.541 %	■	0.713	0.926	0.161	16.870 %	■
0.20	0.407	0.406	0.000	0.115 %	■	0.812	0.803	0.004	0.447 %	■	0.615	0.884	0.122	14.297 %	●
0.50	0.408	0.407	0.000	0.060 %	■	0.812	0.802	0.004	0.472 %	■	0.470	0.929	0.152	16.654 %	●
0.80	0.409	0.408	0.000	0.095 %	■	0.822	0.810	0.004	0.436 %	⋆	0.330	0.660	0.112	17.120 %	
1.00	0.409	0.408	0.000	0.060 %	⋆	0.817	0.808	0.009	1.088 %	○	0.457	0.659	0.277	33.827 %	⋆
	$K = 16$														
	Sharpe index					hypervolume					GD				
P_{EM}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.406	0.405	0.001	0.163 %	■	0.826	0.817	0.006	0.698 %	⋆	0.449	0.690	0.104	14.927 %	
0.05	0.406	0.405	0.001	0.176 %	■	0.828	0.817	0.006	0.762 %		0.402	0.687	0.159	21.543 %	⋆
0.10	0.407	0.405	0.001	0.159 %	■	0.827	0.816	0.005	0.558 %		0.491	0.721	0.139	19.852 %	
0.20	0.408	0.405	0.001	0.134 %	■	0.826	0.816	0.004	0.545 %		0.430	0.693	0.097	13.832 %	
0.50	0.408	0.406	0.001	0.159 %	■	0.823	0.814	0.007	0.819 %		0.449	0.707	0.157	19.944 %	
0.80	0.408	0.407	0.001	0.133 %	■	0.804	0.787	0.006	0.783 %	■	0.804	1.400	0.264	18.547 %	■
1.00	0.408	0.407	0.000	0.069 %	⋆	0.757	0.727	0.010	1.436 %	■	2.716	3.955	0.609	15.549 %	■

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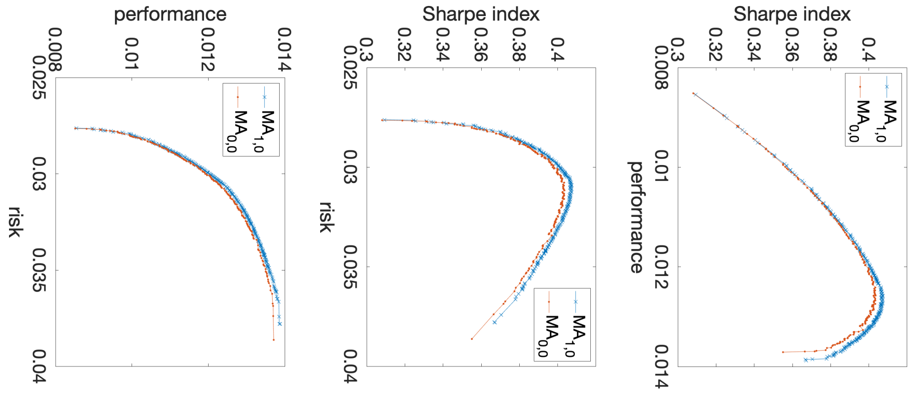
	$K = 18$														
	Sharpe index					hypervolume					GD				
P_{EM}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.404	0.402	0.001	0.150 %	■	0.819	0.802	0.006	0.694 %		0.616	1.149	0.161	13.912 %	
0.05	0.404	0.402	0.001	0.235 %	■	0.813	0.800	0.007	0.927 %		0.757	1.268	0.224	17.349 %	
0.10	0.406	0.402	0.001	0.222 %	■	0.815	0.801	0.004	0.550 %		0.783	1.139	0.130	11.611 %	
0.20	0.406	0.403	0.001	0.181 %	■	0.816	0.804	0.010	1.211 %	⋆	0.672	1.114	0.215	17.980 %	⋆
0.50	0.406	0.404	0.001	0.235 %	■	0.817	0.794	0.005	0.609 %	●	0.569	1.298	0.187	14.047 %	○
0.80	0.407	0.406	0.001	0.137 %	■	0.810	0.769	0.009	1.220 %	■	0.978	2.059	0.434	21.101 %	■
1.00	0.408	0.407	0.001	0.146 %	⋆	0.744	0.704	0.012	1.755 %	■	3.627	5.483	0.695	13.401 %	■
	$K = 20$														
	Sharpe index					hypervolume					GD				
P_{EM}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.398	0.389	0.007	1.714 %	■	0.865	0.820	0.025	3.024 %	■	2.466	3.719	1.138	26.389 %	■
0.05	0.402	0.388	0.011	2.837 %	■	0.866	0.822	0.033	4.020 %	■	1.730	3.953	1.559	39.962 %	■
0.10	0.402	0.392	0.009	2.308 %	■	0.886	0.823	0.033	4.004 %	■	1.268	3.214	1.333	37.176 %	■
0.20	0.404	0.395	0.008	2.051 %	■	0.902	0.840	0.023	2.793 %	○	0.968	2.708	1.084	36.466 %	■
0.50	0.408	0.405	0.005	1.358 %	■	0.910	0.879	0.033	3.872 %	⋆	0.575	1.053	0.547	41.557 %	
0.80	0.409	0.408	0.001	0.300 %	⋆	0.910	0.852	0.031	3.667 %		0.382	0.906	0.607	46.891 %	⋆
1.00	0.409	0.408	0.117	39.895 %		0.904	0.810	0.355	68.767 %	■	0.249	1.919	19.898	96.707 %	●

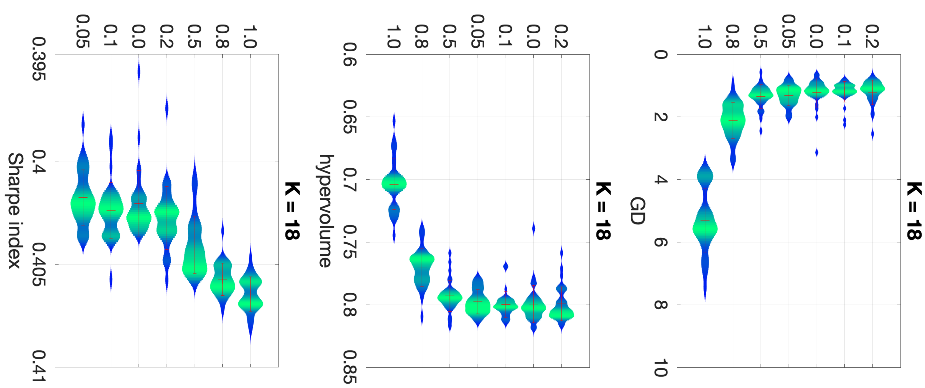
Table 3. Numerical results using elite memory



a) b) c)

Figure 2. Comparison of the combined Pareto fronts attained by $\mathrm{MA}_{1,0}$ and $\mathrm{MA}_{0,0}$ (IBEA) for $K = 18$ a). Sharpe index values for nondominated solutions as a function of risk b) and return c).

either case, it is important to recall that this pressure is exerted in the realm of Sharpe index. Thus, note how high values of P_{EM} (not necessarily the maximum values considered in all cases) result in improved performance with respect to this indicator with high statistical significance with respect to lower values. However, note the opposite effect on hypervolume and generational distance. We attribute these results to the algorithm's settling on suboptimal solutions with respect to the Sharpe index, which can therefore spread over larger regions of the Pareto front. In



a) Sharpe index b) Hypervolume c) Generational distance

Figure 3. Distribution of the performance indicators attained across 30 runs of the MA with different parameterization of elite memory and no local search for $K = 18$

fact, we can observe in Figure 4 how the bulge in the region where the highest values of the Sharpe Index are confined is much more pronounced and indicates how the search focus is very much on that region as opposed to the extremes of the Pareto front, which affects the values of the pure multiobjective performance indicators. Of course, this is a sought behavior, since regions of the Pareto front away from this bulge are increasingly worse in terms of the Sharpe index, and therefore less useful for the subsequent decision-making process. Thus, the seemingly better values of these pure multiobjective indicators are to some extent an artifact of their spread on regions of limited usefulness.

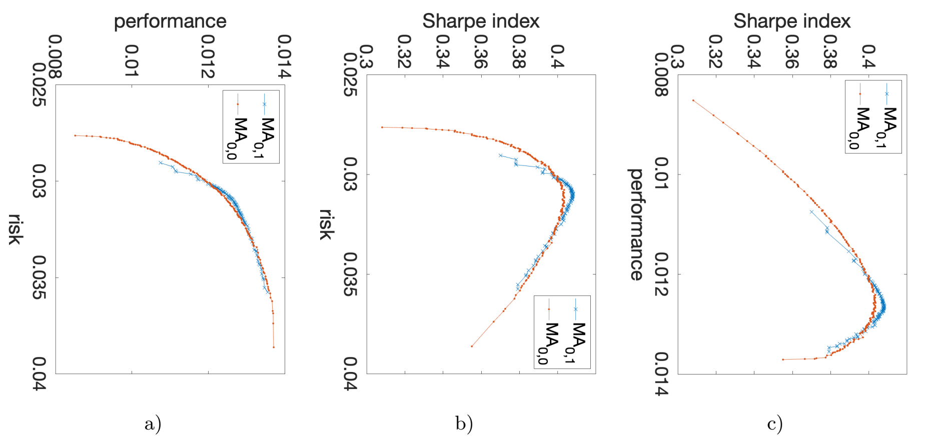


Figure 4. Comparison of the combined Pareto fronts attained by $MA_{0,1}$ and $MA_{0,0}$ (IBEA) for $K = 18$ a). Sharpe index values for nondominated solutions as a function of risk b) and return c).

Local Search and Elite Memory. Having tested the sensibility to each of the two selected components of the MA in the previous section, let us focus on their interplay. We are building on these previous experiments to identify regions of interest for the corresponding parameters in order to avoid a very expensive fully factorial experimentation. More precisely, we have focused on values $P_{EM} \in \{0.0, 0.8, 1.0\}$ and considered $P_{LS} \in \{0.0, 1.0\}$. Recall that this selection has been made attending to Sharpe index as the primary measure of interest, and that among the different parameter combinations attainable here we have both a plain MOEA (namely IBEA), and an MA that features only local search or elite memory. The numerical data are provided in Table 4, and an illustration of the distribution of results for $K = 18$ is provided in Figure 5.

It is clear from these results that a full-throttle MA ($P_{EM} = 1.0, P_{LS} = 1.0$) provides the best results from the point of view of Sharpe index. Although this is not necessarily the best combination with respect to purely multiobjective per-

		$K = 14$														
		Sharpe index					hypervolume					GD				
P_{EM}	P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.00	0.408	0.407	0.000	0.073 %	$\blacksquare$	0.801	0.794	0.002	0.280 %	$\blacksquare$	0.771	1.200	0.167	13.924 %	$\blacksquare$
0.00	1.00	0.409	0.408	0.000	0.051 %	$\blacksquare$	0.809	0.803	0.002	0.201 %	$\star$	0.458	0.916	0.163	17.379 %	$\star$
0.80	0.00	0.409	0.408	0.000	0.095 %	$\blacksquare$	0.809	0.797	0.004	0.449 %	$\blacksquare$	0.545	1.095	0.165	14.300 %	$\blacksquare$
0.80	1.00	0.409	0.409	0.000	0.028 %	$\blacksquare$	0.807	0.772	0.031	4.036 %	$\blacksquare$	0.551	1.355	1.082	53.699 %	$\blacksquare$
1.00	0.00	0.409	0.408	0.000	0.060 %	$\blacksquare$	0.804	0.795	0.008	1.037 %	$\blacksquare$	0.662	1.046	0.281	25.079 %	$\bullet$
1.00	1.00	0.410	0.409	0.000	0.025 %	$\star$	0.750	0.724	0.003	0.453 %	$\blacksquare$	2.265	3.288	0.131	4.033 %	$\blacksquare$
		$K = 16$														
		Sharpe index					hypervolume					GD				
P_{EM}	P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.00	0.406	0.405	0.001	0.163 %	$\blacksquare$	0.806	0.798	0.005	0.657 %	$\blacksquare$	1.168	1.717	0.186	10.722 %	$\blacksquare$
0.00	1.00	0.408	0.407	0.000	0.059 %	$\blacksquare$	0.821	0.816	0.002	0.263 %	$\star$	0.686	0.957	0.154	15.887 %	$\star$
0.80	0.00	0.408	0.407	0.001	0.133 %	$\blacksquare$	0.785	0.770	0.006	0.768 %	$\blacksquare$	1.549	2.705	0.339	13.219 %	$\blacksquare$
0.80	1.00	0.409	0.409	0.000	0.055 %	$\blacksquare$	0.825	0.744	0.027	3.485 %	$\blacksquare$	0.557	2.392	0.872	47.270 %	$\blacksquare$
1.00	0.00	0.408	0.407	0.000	0.069 %	$\blacksquare$	0.741	0.717	0.009	1.299 %	$\blacksquare$	3.176	4.241	0.568	13.007 %	$\blacksquare$
1.00	1.00	0.410	0.409	0.000	0.018 %	$\star$	0.750	0.735	0.003	0.341 %	$\blacksquare$	2.359	2.785	0.169	5.809 %	$\blacksquare$

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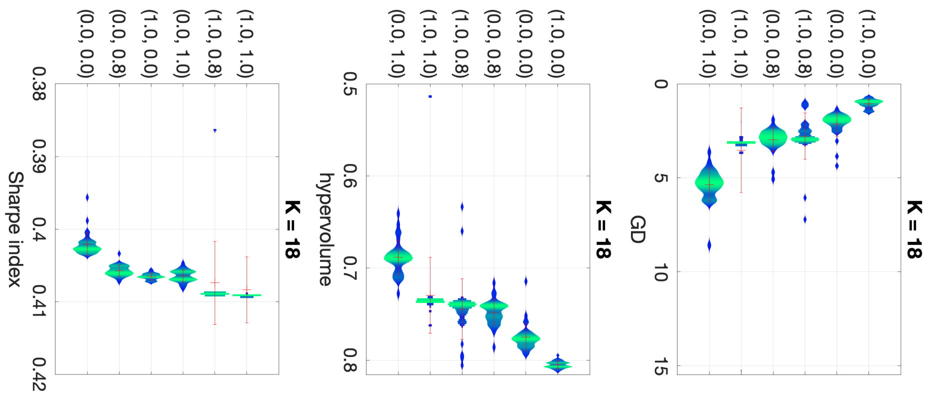
		$K = 18$														
		Sharpe index					hypervolume					GD				
P_{EM}	P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.00	0.404	0.402	0.001	0.150 %	■	0.793	0.777	0.006	0.814 %	■	1.345	1.986	0.273	13.275 %	■
0.00	1.00	0.407	0.407	0.000	0.066 %	■	0.812	0.806	0.002	0.310 %	⋆	0.680	1.003	0.150	14.564 %	⋆
0.80	0.00	0.407	0.406	0.001	0.137 %	■	0.786	0.746	0.008	1.051 %	■	1.896	2.898	0.314	10.905 %	■
0.80	1.00	0.409	0.409	0.000	0.037 %	■	0.806	0.742	0.009	1.201 %	■	0.967	2.859	0.340	12.696 %	■
1.00	0.00	0.408	0.407	0.001	0.146 %	■	0.728	0.690	0.009	1.234 %	■	3.631	5.282	0.402	7.685 %	■
1.00	1.00	0.409	0.409	0.000	0.022 %	⋆	0.763	0.735	0.001	0.178 %	■	1.970	3.155	0.069	2.170 %	■

		$K = 20$														
		Sharpe index					hypervolume					GD				
P_{EM}	P_{LS}	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
0.00	0.00	0.398	0.389	0.007	1.714 %	■	0.876	0.832	0.025	3.001 %	■	2.596	3.787	1.041	24.091 %	■
0.00	1.00	0.406	0.405	0.001	0.156 %	■	0.903	0.889	0.012	1.331 %	⋆	0.772	1.186	0.114	9.894 %	●
0.80	0.00	0.409	0.408	0.001	0.300 %		0.921	0.865	0.031	3.540 %		0.454	0.924	0.441	38.104 %	⋆
0.80	1.00	0.409	0.409	0.000	0.038 %		0.910	0.818	0.006	0.674 %	■	0.555	1.644	0.146	9.081 %	●
1.00	0.00	0.409	0.408	0.117	39.895 %		0.916	0.824	0.358	68.208 %	■	0.258	1.515	19.594	96.556 %	●
1.00	1.00	0.409	0.409	0.090	28.212 %	⋆	0.856	0.811	0.282	52.761 %	■	0.920	1.951	16.353	89.935 %	■

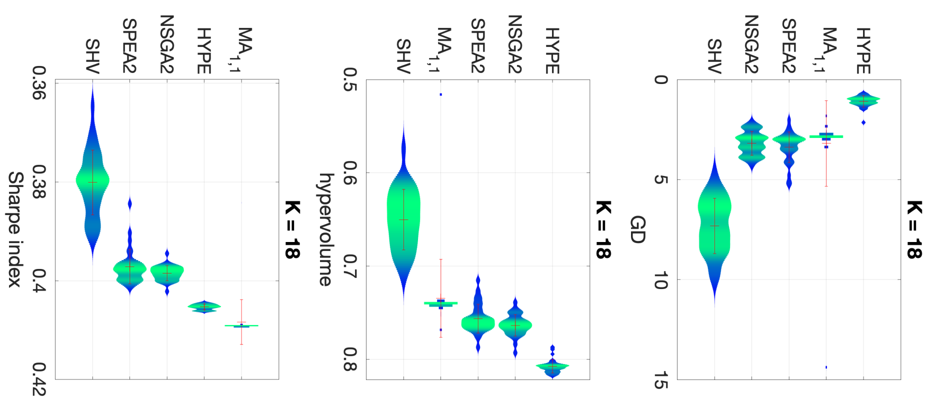
Table 4. Numerical results using elite memory and local search



a) Sharpe index b) Hypervolume c) Generational distance

Figure 5. Distribution of the performance indicators attained across 30 runs of the MA with different parameterization of elite memory and local search for $K = 18$. Each pair of values in the y-axis represent (P_{LS}, P_{EM}).

formance indicators, it seems that it results in the most synergistic interplay of these two components when it comes to finding risk/performance-efficient portfolios. This finding is statistically robust as indicated by the Wilcoxon ranksum test and will be benchmarked against other optimization algorithms in the next subsection.



a) Sharpe index b) Hypervolume c) Generational distance

Figure 6. Distribution of the performance indicators attained across 30 runs of the algorithms (the MA with $P_{LS} = 1.0$ and $P_{EM} = 1.0$, NSGA2, SPEA2, HYPE and SHV)

	$K = 14$														
	Sharpe index					hypervolume					GD				
algorithm	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
$\mathrm{MA}_{1,1}$	0.410	0.409	0.000	0.025 %	⋆	0.750	0.724	0.003	0.457 %	■	2.256	3.271	0.124	3.817 %	■
NSGA2	0.406	0.405	0.001	0.148 %	■	0.786	0.776	0.005	0.697 %	■	1.613	2.411	0.307	13.570 %	■
SPEA2	0.406	0.404	0.001	0.218 %	■	0.789	0.777	0.007	0.841 %	■	1.430	2.145	0.424	19.834 %	■
HYPE	0.408	0.407	0.000	0.057 %	■	0.809	0.802	0.003	0.356 %	⋆	0.463	0.647	0.075	11.401 %	⋆
SHV	0.403	0.399	0.001	0.207 %	■	0.753	0.728	0.010	1.414 %	■	2.579	3.764	0.624	15.373 %	■
	$K = 16$														
	Sharpe index					hypervolume					GD				
algorithm	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
$\mathrm{MA}_{1,1}$	0.410	0.409	0.000	0.018 %	⋆	0.746	0.732	0.003	0.346 %	■	2.663	3.197	0.175	5.286 %	■
NSGA2	0.404	0.402	0.001	0.254 %	■	0.790	0.771	0.005	0.677 %	■	1.473	2.540	0.238	9.204 %	■
SPEA2	0.405	0.402	0.002	0.392 %	■	0.789	0.772	0.009	1.171 %	■	1.577	2.467	0.367	14.940 %	■
HYPE	0.407	0.406	0.000	0.069 %	■	0.812	0.804	0.004	0.559 %	⋆	0.614	0.886	0.126	13.478 %	⋆
SHV	0.396	0.392	0.002	0.463 %	■	0.731	0.695	0.011	1.551 %	■	3.999	5.527	0.813	15.104 %	■

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	$K = 18$														
	Sharpe index					hypervolume					GD				
algorithm	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
$MA_{1,1}$	0.409	0.409	0.000	0.022 %	$\star$	0.769	0.740	0.001	0.171 %	■	1.776	2.834	0.073	2.555 %	■
NSGA2	0.402	0.398	0.001	0.282 %	■	0.793	0.763	0.007	0.893 %	■	2.124	3.202	0.487	14.678 %	■
SPEA2	0.401	0.398	0.002	0.407 %	■	0.787	0.759	0.009	1.194 %	■	2.006	3.157	0.379	11.433 %	■
HYPE	0.406	0.405	0.000	0.115 %	■	0.818	0.808	0.004	0.440 %	$\star$	0.666	1.054	0.150	13.926 %	$\star$
SHV	0.392	0.380	0.004	1.115 %	■	0.712	0.650	0.025	3.763 %	■	4.712	7.439	1.089	14.638 %	■
	$K = 20$														
	Sharpe index					hypervolume					GD				
algorithm	x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD		x^*	$\tilde{x}$	QD	CQD	
$MA_{1,1}$	0.409	0.409	0.090	28.212 %	$\star$	0.857	0.812	0.287	54.223 %	■	0.921	1.828	16.420	90.376 %	■
NSGA2	0.395	0.388	0.006	1.613 %	■	0.872	0.824	0.019	2.346 %	■	2.472	3.856	0.785	19.122 %	■
SPEA2	0.399	0.389	0.009	2.239 %	■	0.867	0.828	0.025	2.992 %	■	2.147	3.679	0.995	25.632 %	■
HYPE	0.406	0.403	0.003	0.766 %	■	0.907	0.884	0.016	1.785 %	$\star$	0.522	1.088	0.675	44.540 %	$\star$
SHV	0.390	0.351	0.010	2.721 %	■	0.804	0.694	0.024	3.427 %	■	3.536	9.504	1.332	13.845 %	■

Table 5. Numerical results for all algorithms considered (the MA with $P_{LS} = 1.0$ and $P_{EM} = 1.0$, NSGA2, SPEA2, HYPE and SHV). Note that IBEA is not included in this table because it is equivalent to $MA_{0,0}$ and this comparison was already made in the sensitivity analysis of Section 4.2.1.

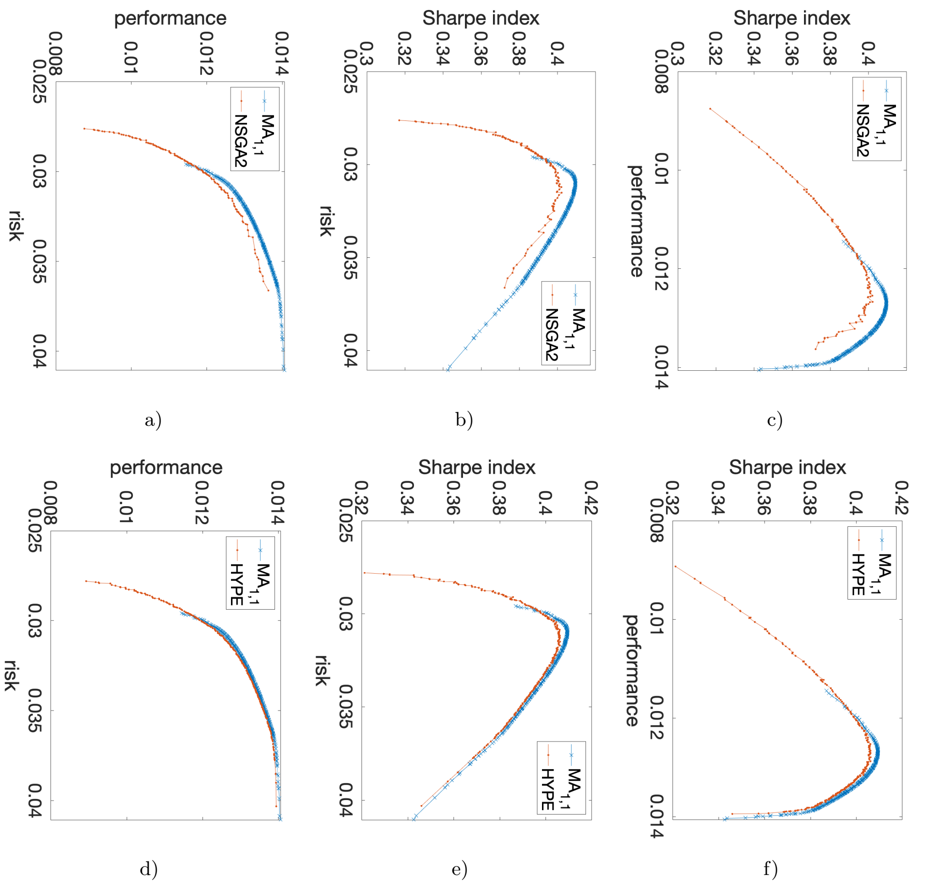

Figure 7. Comparison of the combined Pareto fronts attained by $\mathrm{MA}_{1,1}$ and NSGA2 for $K = 18$ a). Sharpe index values for nondominated solutions as a function of risk b) and return c). Subfigures d)–f) are analogous for HYPE.

4.2.2 Performance Analysis

The experimentation has been carried out with the six algorithms previously described in Section 3.1, namely NSGA-II, SPEA2, HYPE, and SHV (note that the comparison with IBEA – $\mathrm{MA}_{0,0}$ – was already made in the sensitivity analysis), and our memetic algorithm described in Section 3.2, in this case using $P_{EM} = 1.0$ and $P_{LS} = 1.0$. Table 5 provides the numerical results, and Figure 6 contains the relative comparison of the values achieved with respect to Sharpe index, hypervolume and generational distance for $K = 18$ as in previous cases.

When analyzed from the prism of the Sharpe index, which is our primary gauge as a problem-specific financial indicator, the selected $\mathrm{MA}_{1,1}$ is the clear winner, outperforming the rest of algorithms with statistical significance ($\alpha = 0.01$). This is also clear when inspecting Figure 6 a). Note also the very distinctive shape of the Pareto front achieved (Figure 7), and how there is not only a bulge in the immediate region surrounding the highest values of the Sharpe index, but the superiority also extends towards the high-risk region, particularly when compared to NSGA2. Of course, this improvement comes at the cost of sacrificing performance in the low-risk region of portfolios, resulting in lower values of pure multiobjective performance indicators. Then again, if we observe that the region with the largest values of the Sharpe index is the area of interest for containing better excess return per unit of risk, it makes sense to consider better coverage of this region and have the human expert select an appropriate portfolio according to the risk profile sought within this focused subset of highly efficient portfolios.

5 CONCLUSIONS

The portfolio optimization problem is a natural scenario for the use of multiobjective evolutionary algorithms, due to the intrinsic formulation of the problem and to the capacity and flexibility of these types of algorithms. A particularly interesting feature of the problem considered is the existence of a natural indicator of portfolio efficiency (Sharpe index in this case). This has two main advantages: on one hand, it provides an appropriate mechanism for singling out a solution after the multiobjective optimization process (a step that is often overlooked in this kind of scenarios); on the other hand, it constitutes a legitimate measure to indicate preferred search directions in the multiobjective space. In this sense, the absence of such indications is typically one of the difficulties that memetic algorithms face in these kinds of domains, typically causing local search to be either defined randomly chosen directions or augmented to perform an ascent in a Pareto sense (which can also cause it to drift in different directions, not necessarily aligned with financially efficient portfolios in the sense considered here). Exploiting this indicator can provide a reasonable goal to local search, though, hence adapting the search to the problem specificities, as well as aligning with the ulterior decision-making process by the human expert.

Our experimentation with data from 20 securities taken from the Colombian Stock Exchange in the period between 2010 and 2016 has provided experimental support to this claim. The algorithmic model considered includes the use of an elite memory mechanism based on this indicator to modulate the intensification/diversification profile of the search. As shown in the previous section, the results of the MA are markedly better than other multiobjective algorithmic proposals on the described benchmark. This highlights the effective synergy between the algorithmic components the MA is endowed with, and how their interplay helps focus the search towards efficient portfolios.

There are many avenues for further research. Undoubtedly, confirming these findings on other datasets is one of them. We are also particularly interested in the algorithmic issues that stem from decentralizing the population [60], that is, using a non-panmictic model in which multiple MA islands evolve in partial isolation. We believe that this mechanism can be particularly useful in adding an additional layer of diversification/intensification balance in this multiobjective domain. We also envision adaptive procedures to control the usage of MA components – work is underway in this direction [61]. These more sophisticated approaches could also be tested against other mathematical models, such as the Black and Litterman [62] model. We could also mention that although we have focused here on the use of the Sharpe index, as a knowledge-infused indicator that would direct the search towards the regions of interest for the decision-maker, there are some other possibilities that could be considered. For example, machine learning approaches [63] have also shown promising results and can be useful as knowledge providers instead of the Sharpe index. The generic methodological approach presented in this work can be easily adapted to such increasingly sophisticated sources of expert knowledge. We can also consider alternatives to the Markowitz model. The latter is more oriented towards stable markets due to the assumption that historical data offers insight of future returns and risks; studying markets in other scenarios (i.e., expanding markets, volatile markets, receding markets, etc.) may require the use of other models more oriented towards such conditions, e.g., the capital asset pricing model [64] or the arbitrage pricing theory [65], just to mention two examples.

6 STATEMENTS AND DECLARATIONS

Author Contributions. All authors contributed to the conception, design and analysis of the study. The preparation of the material, the collection of data and the preparation of the first draft of the manuscript were carried out by F.C.D. All authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Competing Interests. The authors have no competing interests to declare relevant to the content of this article.

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Feijoo Colomine Duran is an M.Sc. in management with a mention in finance from the National Experimental University of Táchira, Venezuela, Systems Engineer from the University of the Andes. Diploma of Advanced Studies in Computer Technologies from the University of Malaga, he is currently a doctoral student at the same university. His current research interests include artificial intelligence, evolutionary algorithms, and programming. Additionally, he has worked in the public sector as a university professor for more than 25 years.

Carlos Cotta is Professor of computer science at the University of Málaga, Spain. He holds his Ph.D. in computer science from the same university, where he has been a faculty member since 1995. Currently, he co-leads the CAESIUM research group, and his research interests include evolutionary and memetic computation, heuristic optimization, and complex systems, areas in which he has contributed to more than 200 research articles. His work emphasizes both the theoretical foundations of computational methods and their practical applications, helping to bridge the gap between fundamental research and real-world challenges.

Antonio J. Fernández-Leiva is Professor at the University of Málaga. He holds his Ph.D. in computer science and artificial intelligence. He co-heads the CAESIUM research group, and his main research area involves the application of artificial intelligence to optimization problems and video games. He is the Principal Investigator of the Spanish Network of Excellence in R&D and Science in Videogames as well as a member of the Expert Advisory Committee for the design of the Artificial Intelligence Strategy of Andalusia (2023–30). He is (co-)author of dozens of research papers focused on the optimization of combinatorial problems.